TRAVAUX EN COURS

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Diagonal complexes and F-gauge structures

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CONTENTS

INTRODUCTION

There have been two attempts to put extra structure on the crystalline cohomology of a smooth and proper variety in positive characteristic and giving, in particular, a closer relation with the Hodge cohomology. The first such attempt is Mazur's theory of gauge structures, further developped by Ogus. The second is the theory of the de Rham-Witt complex of Bloch-Deligne-Illusie.

It is the purpose of this paper to continue the study of the algebraic properties of these structures begun in [Il-Ra] and [Ma]. It turns out, in fact, that in a very precise sense the two approaches are equivalent. Hence we will study coherent R-modules, where R is the Raynaud ring, and coherent F-gauge structures, notions which capture the essential algebraic properties of the cohomology of the de Rham-Witt complex and crystalline cohomology with its gauge structure respectively.

However, it is not the notions of coherent F-gauge structures and coherent R-modules which are equivalent, it is the categories of bounded coherent complexes of F-gauge structures and bounded coherent R-complexes that are and the equivalence we will in fact construct will not take coherent F-gauge structures to coherent R-modules nor vice versa. This is one of the reasons why we will consistently work with derived categories. Another reason is that a complex contains more information than its cohomology. A striking example of this is the case of the cohomology of a supersingular K3-surface with Artin invariant σ. The cohomology R-modules of the de Rham-Witt complex depend only on σ, whereas the whole complex determines the periods up to finite indeterminacy and so has $\sigma-1$ moduli.

Let me give a more detailed description of the contents of the present paper.

If we think of the canonical truncations of a complex as giving a filtration of that complex, what the first chapter is concerned with is the definition and study of another filtration on coherent R-complexes. More precisely, in the terminology of [B-B-D] which is eminently suited for our purposes, we will define a non-standard t-structure on the triangulated category of coherent R-complexes. This t-structure will be called the diagonal t-structure. The reason for this is that it filters a coherent R-complex M diagonally ; $\tilde{\tau}_{\leq i}M/\tilde{\tau}_{\leq i-1}M$ has the property that its j:th cohomology module is the part of $H^j(M)$ generated by degrees $\leq i-j$ divided by the part generated by degrees $\leq i-j-1$ and so can be said to be the part of M in total degree i where a complex of R-modules is thought of as a double complex. From this point of view the standard t-structure is the horizontal t-structure. It should be pointed out that although the diagonal t-structure is characterized by the description just given of $\tilde{\tau}_{\leq i}M/\tilde{\tau}_{\leq i-1}M$ its existence is a non-trivial fact. If we consider instead $D^b_c(k[d])$ where k is a field and d is of degree 1 then the diagonal t-structure does not exist. We will show the existence of our diagonal t-structure with the aid of a general criterion which may be of independent interest.

Using the general theory of [B-B-D] we then see that the coherent R-complexes concentrated in total degree 0, the diagonal complexes, form an abelian category which we will denote Δ. Much of my effort has been spent on a study of this category. The first result we obtain is that Δ is a noetherian category. This is surprising as the category of coherent R-modules is not. The strategy to be used in the further study of Δ is then the following. If we have a subcategory M of Δ we try to find two subcategories M_1 and M_2 of M such that every object of M is canonically an extension of an object of M_1 by an object of M_2 and any such extension is isomorphic to the canonical extension of the middle object of that extension. If we can do this then the classification of objects of M

is reduced, in principle, to the classification of objects of M_1 and M_2 and the computations of the action of automorphism groups on Ext^1-groups.

The first example of this is the presentation of any diagonal complex as the extension of a finite torsion complex by one without finite torsion, where a diagonal complex is finite torsion if it is a sum of complexes of the form $M(i)[-i]$ for different i with M an R-module concentrated in degree 0 of finite length as W-module and a diagonal complex is without finite torsion if it contains no finite torsion subobject. The category of finite torsion objects is as known as the category of finite length Dieudonné modules is and we direct our attention to diagonal complexes without finite torsion.

The crucial theorem I:4.5 then, roughly speaking, allows us to consider only the cases of diagonal dominoes and Mazur-Ogus diagonal complexes.

The diagonal dominoes are the diagonal complexes which are at the same time p-torsion and without finite torsion. Using again (I:4.5) we define a $1/2\mathbb{Z}$-filtration $0 \leq \dots W^i(-) \leq W^{i-1/2}(-) \leq W^{i-1}(-) \leq \dots \leq id$ on the category of diagonal dominoes and we will say that a diagonal domino M is of type i if $W^iM = M$ and $W^{i+1/2}M = 0$. The category of diagonal dominoes of type i is equivalent to the category of diagonal dominoes of type $i+n$ for any $n \in \mathbb{Z}$. The diagonal dominoes of type 0 are exactly those diagonal complexes whose associated simple complex is acyclic. We will also show that this category is equivalent to the category of graded W-modules $M = \bigoplus_{i\in\mathbb{Z}} M(i)$ of finite length together with W-endomorphisms $\widetilde{F}$ and $\widetilde{V}$ of degree 1 and -1 s.t. $\widetilde{F}\widetilde{V} = \widetilde{V}\widetilde{F} = p$ where p is the characteristic of our basefield. This description allows us in particular to describe all indecomposable type 0 diagonal dominoes killed by p. In the case of halfinteger type diagonal dominoes the results are less complete. We will however give a number of type $-1/2$ diagonal dominoes having the property that every type $-1/2$ diagonal domino is a successive extension of some of the given ones and none of the given ones are non-trivial extensions of type $-1/2$ diagonal dominoes. We

will also give a description of type $-1/2$ diagonal dominoes similar, but less convenient, to the one given of type 0 diagonal dominoes.

It should be noted that type 0 diagonal dominoes form an abelian category but the type $-1/2$ diagonal dominoes do not.

The Mazur-Ogus diagonal complexes are precisely those diagonal complexes such that the rank of the associated simple complex equals the sum of the Hodge numbers, where the Hodge numbers are defined in analogy with the geometric case. This is exactly the condition considered by Mazur and Ogus hence justifying the notation. The associated simple complex is then a free (cf. [B-O]) W-module of finite rank and can in a natural fashion be given the structure of an F-crystal (rather a virtual F-crystal). This association of a (virtual) F-crystal to a Mazur-Ogus diagonal complex turns out to be an equivalence of categories and an explicit inverse is described. A rather direct consequence of this is the Mazur description of the Hodge filtration in terms of the F-crystal structure. Thus our result is seen as a generalization of the Mazur-Ogus result ; not only can the Hodge filtration be explicitly recovered from the crystalline cohomology but so can the whole Hodge-Witt cohomology.

An F-gauge structure is a graded W-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ together with linear mappings $\widetilde{F}$ and $\widetilde{V}$ of degree 1 resp. -1 and a σ-linear isomorphism $\tau : M^{\infty} := \varinjlim(\ldots \xrightarrow{\widetilde{F}} M^i \xrightarrow{\widetilde{F}} M^{i+1} \xrightarrow{\widetilde{F}} \ldots) \longrightarrow M^{-\infty} := \varinjlim(\ldots \xrightarrow{\widetilde{V}} M^i \xrightarrow{\widetilde{V}} M^{i-1} \xrightarrow{\widetilde{V}} \ldots)$, M is said to be coherent if M^i is of finite type as W-module for all i and $\widetilde{V} : M^i \longrightarrow M^{i-1}$ (resp. $\widetilde{F} : M^i \longrightarrow M^{i+1}$) are isomorphisms for all $i \ll 0$ (resp. $i \gg 0$).

To every complex M of R-modules we may associate a complex of F-gauge structures $\underline{\underline{S}}(M)$ where $\underline{\underline{S}}(M)^{\infty} = \underline{\underline{s}}(M)$, the associated simple complex and the $\underline{\underline{S}}(M)^i$ are the simple complex associated to suitable modifications à la Nygaard of M. One of the main results of this paper then says that $\underline{\underline{S}}(-)$ is an equivalence of categories from bounded coherent R-complexes to

bounded complexes of F-gauge structures with coherent cohomology. It should be noted that this equivalence is used in proving the results on diagonal dominoes and Mazur-Ogus complexes mentioned above which explains why the results in the paper are given in a different order than they are presented in this introduction.

The reader will no doubt notice that this paper is concerned mainly with coherent R-complexes and their diagonal and not F-gauge decomposition, most of the results are obtained using the diagonal decomposition. One of the reasons for this is "historical" ; I discovered the diagonal decomposition first and the F-gauge structures appeared at the very end of my work when I tried to prove that every F-crystal was of the form $\underline{\underline{s}}(M)$, for M a Mazur-Ogus diagonal complex. It seems that F-gauge structures are a perfectly natural generalization of Dieudonné modules, indeed they have been introduced as such by Fontaine independently of the relation with R-modules, and many results obtained here could no doubt be obtained by starting from F-gauge structures rather than R-modules. There is, however, one important case where this seems to be not true. By modification à la Nygaard at different degrees we get an action of the group of functions $\mathbb{Z} \longrightarrow \mathbb{Z}$ with finite support on R-modules which induces an action on bounded coherent R-complexes. This action preserves diagonal complexes but not F-gauge structures, in fact the diagonal t-structure is in some sense the t-structure closest to the F-gauge structure t-structure preserved by this action. Hence the category of diagonal complexes has a large degree of symmetry which the category of coherent F-gauge structures lacks. The filtration by type on diagonal dominoes mentioned above, for instance, is obtained by applying elements of this group to the filtrations of (I:4.5).

On the other hand, F-gauge structures are more concrete objects and so easier to construct. An example of this is the case of coherent F-gauge structures M with $M^{\infty} = 0$ which correspond to type 0 diagonal dominoes. Even for simple examples of such F-gauge structures the corresponding

R-complex can be difficult to understand. Consider for instance the case of such F-gauge structures which are indecomposable and killed by p. As will be seen these are extremely easy to describe and it is easy to see that the corresponding R-complexes have the property that their diagonal cohomology R-complexes have the property that their cohomology R-modules are either zero or isomorphic to a degree shift of one of the $\underline{\underline{U}}_j$:s, using the terminology of [Il-Ra], but except for some simple cases I do not know how to determine the j:s in question.

In chapter IV we will introduce the Hodge-Witt numbers, $h_W^{i,j}$, associated to a coherent R-complex M. They depend only on the slopes of the F-isocrystals $H^i(\underline{\underline{s}}(M)) \otimes_W K$, $K :=$ field of fractions of W, and the $T^{i,j}$ of [Il-Ra] which measure the sizes of the non-finitely generated, as W-modules, parts of $H^j(M)$. From the definition of the $h_W^{i,j}$ it follows that $\sum_{i+j=n} h_W^{i,j} = b_n :=$ the rank of $H^i(\underline{\underline{s}}(M))$ and a formula of Crew is (cf.[Cr]) equivalent to $\sum_j (-1)^j h_W^{i,j} \doteq \sum_j (-1)^j h^{i,j}$. The main result concerning the Hodge-Witt numbers is the inequality $h_W^{i,j} \leq h^{i,j}$. As an immediate consequence we get that if M fulfills the conditions of Mazur and Ogus then $h_W^{i,j} = h^{i,j}$ and so the $T^{i,j}$ can be explicitly determined by the slopes and the Hodge numbers. In particular we get a criterion for when the $T^{i,j}$ are zero. Another consequence is the Katz conjecture on the relation between the Newton and Hodge polygons ; by definition the Newton polygon lies above the Hodge-Witt polygon, the Hodge polygon associated to the Hodge-Witt numbers, and the inequality $h_W^{i,j} \leq h^{i,j}$ is more than enough to ensure that the Hodge polygon lie below the Hodge-Witt polygon.

The main results of this paper are purely algebraic even though they have immediate consequences for the Hodge-Witt cohomology of a smooth and proper variety. I have included, however, some geometric applications and examples. Notably, I show that the generic Zariski surfaces have lots of non-closed 1-forms, give some new examples of closed, not indefinitely

closed, forms and give a discussion of the special properties of the Hodge-Witt numbers in the geometric case.

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Finally, it seems altogether appropriate to dedicate this paper to J. Dieudonné.

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Préliminaires and notations

1. Recall (cf. [B-B-D]) that a t-structure on a triangulated category D is a pair $(D^{\leq 0}, D^{\geq 0})$ of strictly full subcategories of D such that, where $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$.

i) For $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$ $\mathrm{Hom}(X,Y) = 0$.

ii) $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 0} \supseteq D^{\geq 1}$.

iii) $\forall X \in D$ there is a distinguished triangle (A,X,B) with $A \in D^{\leq 0}$, $B \in D^{\geq 1}$.

We will only consider t-structures which are non-degenerate and locally of finite amplitude i.e.

A) $\bigcap_n D^{\leq n} = \bigcap_n D^{\geq n} = \{0\}$.

B) $\forall X \in D$ there is an $n \in \mathbb{N}$ s.t. $X \in D^{\leq n} \cap D^{\geq -n}$.

Remark : i) Hence the standard example will be $D^b(A)$ for A an abelian category rather than $D(A)$ as in [B-B-D].

ii) It is clear that to specify a t-structure it is sufficient to specify the abelian category $D^0 := D^{\leq 0} \cap D^{\geq 0}$ in D and we will sometimes do just that.

If D and E are triangulated categories and $T : D \longrightarrow E$ a functor we say that T is triangulated if there is given a natural equivalence $T(-[1]) \xrightarrow{\sim} T(-)[1]$ and if T takes distinguished triangles to distinguished triangles. If D and E are given t-structures we say that a triangulated functor $T : D \longrightarrow E$ has (D^0, E^0)-amplitude, or simply

D^0-amplitude if no confusion ensues, $[m,n]$ if $T(D^0) \leq E^{[m,n]} := E^{\geq m} \cap E^{\leq n}$ $m \leq n \in \mathbb{Z} \cup \{\infty,-\infty\}$ we say that T is (D^0,E^0)-exact (to the left, to the right), or simply D^0-exact (etc.), if it has (D^0,E^0)-amplitude $[0,0]$ ($[0,\infty[$ resp. $]-\infty,0]$). If (D,D^0) is $(D^b(A)$, standard t-structure) we will sometimes speak of amplitude resp. exactness.

Remark : This terminology differs from that of [B-B-D] where exact is used instead of triangulated and t-exact instead of (D^0,E^0)-exact. The term t-exact can be confusing when one, as we will, considers several different t-structures on the same triangulated category, and our use of the term exact conforms with the use of exact in the theory of abelian categories.

Recall that if A is an abelian category then an idempotent radical on A is a subfunctor $T(-)$ of the identity s.t.

i) $T(T(-)) = T(-)$

ii) $T(id/T(-)) = 0$.

It is clear that this is equivalent to

$$0 = \mathrm{Hom}(T(-), id/T(-)) : A^{op} \times A \longrightarrow Ab .$$

The following lemma is also obvious

Lemma 1.1. *Let* $B \subseteq A$ *be a full subcategory closed under finite sums, extensions and quotients. If* A *is noetherian then for every* $M \in A$ *there is a largest subobject* $T(M) \hookrightarrow M$ *in* B *and* $T(-)$ *is an idempotent radical.*

Definition 1.2. *Let* (D,D^0) *be a triangulated category with a* t*-structure. A radical filtration on* (D,D^0) *is an increasing sequence* $\ldots \subseteq T^i \subseteq T^{i+1} \subseteq T^{i+2} \subseteq \ldots \subseteq id$ *of subfunctors of the identity on* D^0 *s.t.*

i) *For every* $M \in D^0$ $T^i(M) = 0$ *if* $i \ll 0$ *and* $T^i(M) = M$ *if* $i \gg 0$.

ii) $\mathrm{Hom}_D^{j-i}(T^{-j}(M), id/T^{-i}(N)) = 0$ *for all* i,j *and* $M,N \in D^0$.

iii) $\mathrm{Hom}_D^{j-i+1}(T^{-j}(M), id/T^{-i}(N)) = 0$ *for all* $i < j$ *and* $M,N \in D^0$.

Putting $S^i(-) := T^i(-)/T^{i-1}(-)$ ii) *and* iii) *together is, by dévissage, easily shown to be equivalent to*
ii)' $\mathrm{Hom}^r_D(S^j(M),S^i(N)) = 0$ *if* $r \leq i-j-1$ *or* $2 \leq r \leq i-j$ *and* $M,N \in D^0$.

If $T^i(-) = 0$ for $i < m$ and $T^i(-) = \mathrm{id}$ for $i \geq n$ then we say that the radical filtration has amplitude $[m,n]$.

Remark : i) A radical filtration of amplitude $[-1,0]$ is, of course, determined by $T^{-1}(-)$ and ii) and iii) say precisely that $T^{-1}(-)$ is an idempotent radical and we will identify idempotent radicals with radical filtrations of amplitude $[-1,0]$. In general, ii) for $i = j$ gives that $T^i(-)$ is an idempotent radical for all i .

ii) If the radical filtration is of amplitude $[m,n]$ with $n-m \leq 1$ then the conditions i) - ii) only depend on D^0 and not on its realization as the heart of a t-structure. In general, however, these conditions do depend on this realization.

Definition 1.3. *Let* D *be a triangulated category. Two* t-*structures* (D,D^0) *and* $(D,\tilde{D}^0)$ *on* D *are said to commute if* $\tau_{\leq n}\tilde{\tau}_{\leq m} = \tilde{\tau}_{\leq m}\tau_{\leq n}$, $\tau_{\leq n}\tilde{\tau}_{\geq m} = \tilde{\tau}_{\geq m}\tau_{\leq n}$, $\tau_{\geq n}\tilde{\tau}_{\leq m} = \tilde{\tau}_{\leq m}\tau_{\geq n}$ *and* $\tau_{\geq n}\tilde{\tau}_{\geq m} = \tilde{\tau}_{\geq m}\tau_{\geq n}$ *where* $\tau_{\leq m},\tau_{\geq m}$ *resp.* $\tilde{\tau}_{\leq m},\tilde{\tau}_{\geq m}$ *are the truncation functors for* D^0 *resp.* $\tilde{D}^0$.

Theorem 1.4. *Let* (D,D^0) *be a* t-*structure on the triangulated category* D.

i) *If* $\ldots \subseteq T^i(-) \subseteq T^{i+1}(-) \subseteq \ldots \subseteq \mathrm{id}$ *is a radical filtration on* (D,D^0) *then* $\tilde{D}^{\leq 0} = \{X \in D : T^{-i}(H^i(X)) = H^i(X)\ \forall i\}$ *and* $\tilde{D}^{\geq 1} = \{X \in D : \mathrm{id}/T^{-i}(H^i(X)) = H^i(X)\ \forall i\}$ *define a* t-*structure on* D *commuting with* (D,D^0).

ii) *If* $(D,\tilde{D}^0)$ *is a* t-*structure on* D *commuting with* (D,D^0) *then* $\ldots \subseteq \tilde{\tau}_{\leq i} \subseteq \tilde{\tau}_{\leq i+1} \subseteq \ldots \subseteq \mathrm{id}$ *is a radical filtration on* (D,D^0).

iii) *These two constructions are natural inverses to each other.*

Proof : To prove that $(\widetilde{D}^{\leq 0},\widetilde{D}^{\geq 1})$ defines a t-structure we begin by verifying that $\mathrm{Hom}(X,Y)=0$ if $X\in\widetilde{D}^{\leq 0}$ and $Y\in\widetilde{D}^{\geq 1}$. By dévissage we may assume that $X=T^{i}(R)[i]$, $Y=\mathrm{id}/T^{j}(S)[j]$ $R,S\in D^{0}$ so $\mathrm{Hom}(X,Y)=$ $\mathrm{Hom}^{j-i}(T^{i}(R),\mathrm{id}/T^{j}(S))=0$ by (1.2i). Let now $X\in D$. We want to find $Y\in\widetilde{D}^{\leq 0}$, $Z\in\widetilde{D}^{\geq 1}$ and a distinguished triangle $\longrightarrow Y\longrightarrow X\longrightarrow Z\longrightarrow Y[1]\longrightarrow$. We will construct Y and Z by induction on the length $n-m$ of the D^{0}-amplitude $[m,n]$ of X and we will also prove that the Y and Z have D^{0}-amplitude $[m,n]$ which will then give the conditions for commutation as $\tau_{\leq 0}\widetilde{D}^{\leq m}\subseteq\widetilde{D}^{\leq m}$ is clear by definition. If the length is zero then $\longrightarrow T^{-m}(X[m])[-m]\longrightarrow X\longrightarrow \mathrm{id}/T^{-m}(X[m])[-m]\longrightarrow$ has the required properties. Let $i,i+1\in[m,n]$. By hypothesis we may find $Y',Y''\in\widetilde{D}^{\leq 0}$ and $Z',Z''\in\widetilde{D}^{\geq 1}$ and distinguished triangles $\longrightarrow Y'\longrightarrow \tau_{\leq i}X\longrightarrow Z'\longrightarrow$ and $\longrightarrow Y''\longrightarrow$ $\tau_{\geq i+1}X\longrightarrow Z''\longrightarrow$. This gives us a diagram

$$\begin{tikzcd}
Y''[-1] \arrow[d] & Y' \arrow[d] \\
\tau_{\geq i+1}X[-1] \arrow[r] \arrow[d] & \tau_{\leq i}X \arrow[d] \\
Z''[-1] & Z'
\end{tikzcd}$$

I claim that the composite $Y''[-1]\longrightarrow Z'$ is zero and indeed that $\mathrm{Hom}(Y''[-1],Z')=0$. By the induction assumption $Y''\in D^{\geq i+1}\cap\widetilde{D}^{\leq 0}$ and $Z'\in D^{\leq i}\cap\widetilde{D}^{\geq 1}$ so by dévissage we may assume that $Y''=T^{-j}(M)[-j]$, $j\geq i+1$ and $Z'=\mathrm{id}/T^{-k}(N)[-k]$, $k\leq i$ so $\mathrm{Hom}(Y''[-1],Z')=$ $\mathrm{Hom}^{j-k+1}(T^{-j}(M),\mathrm{id}/T^{-k}(N))=0$ by (1.2 ii) as $j\geq i+1>i\geq k$. Hence there is a morphism $Y''[-1]\longrightarrow Z'$ making commutative the following diagram

$$\begin{tikzcd}
Y''[-1] \arrow[r] \arrow[d] & Y' \arrow[d] \\
\tau_{\geq i+1}X \arrow[r] & \tau_{\leq i}X
\end{tikzcd}.$$

By [B-B-D : Prop. 1.1.11] there is a diagram commutative up to sign of distinguished triangles

$$\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & Y''[-1] & \to & Y' & \to & Y & \to & Y'' & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & \tau_{\geq i+1}X[-1] & \to & \tau_{\leq i}X & \to & X & \to & \tau_{\geq i+1}X & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & Z''\ -1 & \to & Z' & \to & Z & \to & Z'' & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

I claim that $Y \in \tilde{D}^{\leq 0}$ and $Z \in \tilde{D}^{\geq 1}$. Indeed, $Y' \in D^{\leq i}$ and $Y'' \in D^{\geq i+1}$ so $H^j(Y) = H^j(Y')$ or $H^j(Y'')$ depending on whether $j \leq i$ or $j > i$ and as $Y', Y'' \in \tilde{D}^{\leq 0}$ we see that $Y \in \tilde{D}^{\leq 0}$ and similarly $Z \in \tilde{D}^{\geq 1}$. Finally, $Y', Z' \in D^{[m,i]}$ and $Y'', Z'' \in D^{[i+1,n]}$ so it is clear that $\tilde{\tau}_{\leq 0}\tau_{\leq i}X = Y' = \tau_{\leq i}\tilde{\tau}_{\leq 0}$; $\tau_{\geq i+1}\tilde{\tau}_{\leq 0}X = Y'' = \tilde{\tau}_{\leq 0}\tau_{\geq i+1}X$; $\tilde{\tau}_{\geq 1}\tau_{\leq i}X = Z' = \tau_{\leq i}\tilde{\tau}_{\geq 1}X$ and $\tilde{\tau}_{\leq 1}\tau_{\geq i+1}X = Z'' = \tau_{\geq i+1}\tilde{\tau}_{\geq 1}X$ and by shifting we get that (D, D^0) and $(D, \tilde{D}^0)$ commute. Conversely if $(D, \tilde{D}^0)$ commutes with (D, D^0) then clearly $\tilde{\tau}_{\leq i}$ and $\tilde{\tau}_{\geq i}$ commute with $H^j(-)$ so that $X \in \tilde{D}^{\leq 0}$ iff $\tilde{\tau}_{\leq 0}(H^i(X)[-i]) = H^i(X)[-i]$ for all i or equivalently $\tilde{\tau}_{\leq -i}(H^i(X)) = H^i(X)$ and similarly $X \in \tilde{D}^{\geq 1}$ iff $\tilde{\tau}_{\geq i+1}H^i(X) = H^i(X)$. Now as $H^j(-)$ commutes with $\tilde{\tau}_{\leq i}$ and $\tilde{\tau}_{\geq i+1}$ we see that if $X \in D^U$ then $\to \tilde{\tau}_{\leq i}X \to X \to \tilde{\tau}_{\geq i+1}X \to$ is an exact sequence in D^0. It therefore only remains to show (1.2 i)-iii)) for $\ldots \subseteq \tau_{\leq i}(-) \subseteq \ldots \subseteq \mathrm{id}$ on D^0. Clearly i) is true because $\tilde{D}^0$ is locally of finite amplitude and ii) follows from the condition $\mathrm{Hom}(X,Y) = 0$ if $X \in \tilde{D}^{\leq 0}$ and $Y \in \tilde{D}^{\geq 1}$. For iii) it is clear that we need to prove that if $X \in D^{\geq i+1} \cap \tilde{D}^{\leq 0}$ and $Y \in D^{\leq i} \cap \tilde{D}^{\geq 1}$ then $\mathrm{Hom}^1(X,Y) = 0$. Let $X[-1] \to Y$ be

any morphism and let $X[-1] \longrightarrow Y \longrightarrow Z \longrightarrow X$ be a distinguished triangle. We then get a diagram commutative up to sign of distinguished triangles from [B-B-D ; Prop. 1.1.11] and the commutation of the $\tau_{\leq n}$, $\tau_{\geq n}$, $\tilde{\tau}_{\leq m}$ and $\tilde{\tau}_{\geq m}$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \tilde{\tau}_{\leq 0}X[-1] \arrow[r] \arrow[d, "\wr"] & 0 \arrow[r] \arrow[d] & \tilde{\tau}_{\leq 0}Z \arrow[r] \arrow[d] & \tilde{\tau}_{\leq 0}X \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & X[-1] \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "\wr"] & Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & \tilde{\tau}_{\geq 1}Y \arrow[r] \arrow[d] & \tilde{\tau}_{\geq 1}Z \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

and the commutativity gives that $X[-1] \longrightarrow Y$ equals zero.

Let us recall that given a t-structure (D, D^0) and a cohomological functor $T : D \longrightarrow A$ then, putting $T^i(-) := T(-[i])$, we get for all $X \in D$ a spectral sequence

$$E_2^{i,j} = T^j(H^i(X)) \Longrightarrow T^{i+j}(X) . \tag{1.5}$$

By an exact category we will always mean an exact category in the sense of Quillen (cf. [Q]). We will say that an exact category is weakly noetherian (artinian) if for every object M every increasing (decreasing) sequence of admissible subobjects is eventually constant. An object M is weakly simple if it does not contain any non-trivial admissible subobjects. It is clear that every object in a weakly noetherian and artinian exact category is a successive extension of weakly simple objects.

<u>Remark</u> : There is no Jordan-Hölder theorem though, as the example of the category of finitely generated projective modules over a non-principal ideal

Dedekind ring shows.

2. Recall (cf. [Il-Ra]) that the Raynaud ring, R, is the graded W-ring, $W := W(k)$, k a perfect field of char. p, generated by F,V in degree 0 and d in degree 1 with relations

$$\begin{array}{llll} FV = VF = p & Fa = A^{\sigma}F & V = Va^{\sigma} & \\ a \in W & da = ad & FdV = d & d^2 = 0 \end{array} \qquad a \in W$$

where $(-)^{\sigma}$ is the Frobenius automorphism of W. Then R is the direct sum

$$\sum_{i\geq 0} WF^i \oplus \sum_{i>0} WV^i \oplus \sum_{i\geq 0} WF^i d \oplus \sum_{i>0} WdV^i .$$

All modules over R will be graded. The degree 0 part, R^0, of R is the Dieudonné ring $W_{\sigma}[F,V]$ with $FV = VF = p$. We put $\hat{R} := \varprojlim R/dV^nR+V^nR$. Even though dV^nR+V^nR is not a two-sided ideal the multiplication of R extends by continuity to $\hat{R}$ which thus becomes a graded ring with $\hat{R}^0 = W_{\sigma}[[V]][F]$. Again all $\hat{R}$-modules will be graded.

Recall also that if I is an interval in $\mathbb{Z}$ (i.e. if $i \leq j \leq k$ and $i,k \in I$ then $j \in I$) we say that an R-module M is of level I if $M^i = 0$ when $i \notin I$ and when $i \in I$ and $j \notin I$, $j > i$, then F is bijective on M^i.

Let φ be a function $\mathbb{Z} \longrightarrow \mathbb{Z}$ with finite support and let M be an R-module. We define a new R-module $M(\varphi)$ as follows $M(\varphi)^i = \sigma_*^s M^i$ where $s = -\sum_{i \leq s+1} \varphi(i)$, F and V are the same and $d : M(\varphi)^{i-1} \longrightarrow M(\varphi)^i$ equals $F^{\varphi(i)}d$ where, as usual $F^{-s}d := dV^s$ for $s > 0$. It is easily verified that this is an R-module and then $(-)(\varphi)$ is an exact functor. Note that $((-)(\varphi))(\psi) = (-)(\varphi+\psi)$ where $\varphi+\psi$ is defined by pointwise addition and $(-)(0) = \mathrm{id}$. Hence the group of functions $\mathbb{Z} \longrightarrow \mathbb{Z}$ of finite support acts on the category of R-modules. Finally, $(-)(\varphi)$ passes trivially to the derived category.

Recall further that R_n is the right R-module R/dV^nR+V^nR (so that $\hat{R} := \varprojlim R_n$) and that for any complex M of R-modules we define its completion as $\hat{M} := R\varprojlim(R_n \otimes^L_R M)$. Then (cf. [Ek 1]) M is said to be complete if the natural morphism $M \longrightarrow \hat{M}$ is an isomorphism in $D(R)$, and we always have $\hat{\hat{M}} = \hat{M}$.

Proposition 2.1. *Let* $M \in D(R)$. *Then* $\hat{M}(\varphi) = \widehat{M(\varphi)}$ *for every* $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ *with finite support.*

Proof : By the additivity $(-)(\varphi)(\psi) = (-)(\varphi+\psi)$ we reduce to $\varphi(n) = \delta_{in}$ or $-\delta_{in}$ for some i . By dévissage and resolution we reduce to M a free R-module and then the only non-trivial case is when M has all its generators in degree $i-1$. We then easily reduce to $M = R(-i)$ and compute.

3. Recall that $\underline{\underline{s}} : D(R) \longrightarrow D(W)$ denotes the simple associated complex where a complex of R-modules is considered as a double complex through d and the ordinary differential. We then have (cf. [Ek 1 : III.5.6.1])

$$\underline{\underline{s}}(R_1 \otimes^L_R (-)) = k \otimes^L_W \underline{\underline{s}}(-) \ . \tag{3.1}$$

In [Ek 2] an internal Hom-functor $R\mathrm{Hom}^!_R(-,-)$ is defined on $D(R)^{op} \times D^+(R)$. It has the properties (loc. cit.) that

$$\begin{aligned} &R\mathrm{Hom}_R(-,-) = R\mathrm{Hom}_R(W, R\mathrm{Hom}^!_R(-,-)) = R\mathrm{Hom}^!_R(-,-)^F := \\ &\mathrm{Cone}(F-1 : \ R\mathrm{hom}^!_R(-,-) \longrightarrow R\mathrm{Hom}^!_R(-,-)) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\underline{\underline{s}}(R\mathrm{Hom}^!_R(-,-)) = R\mathrm{Hom}_W(\underline{\underline{s}}(-), \underline{\underline{s}}(-)) \\ &R_1 \otimes^L_R R\mathrm{Hom}^!_R(-,-) = R\mathrm{Hom}_R(R_1 \otimes^L_R(-), R_1 \otimes^L_R (-)) \ . \end{aligned} \tag{3.3}$$

and

$$(3.3.1)\qquad \begin{aligned} R_1 \otimes^{L}_{R} ((-)\hat{\star}^{L}_{R}(-)) &= R_1 \otimes^{L}_{R} (-) \otimes_k R_1 \otimes^{L}_{R} (-) \\ \underline{\underline{s}}((-)\hat{\star}^{L}_{R}(-)) &= \underline{\underline{s}}(-) \otimes^{L}_{W} \underline{\underline{s}}(-) \end{aligned}$$

where, (loc. cit.), $(-)\hat{\star}^{L}_{R}(-)$ is the companion internal tensor product.

Recall also that one puts $D(-) := \mathrm{RHom}^{!}_{R}(-,W)$ and then we have $D(D(-)) = \mathrm{id}$ on $D^b_c(R)$.

In [Ek 2] right R-modules Z_n, B_n are defined, Z_1 and B_1 being the kernel and image respectively of left multiplication by d on R_1 and Z_n and B_n being defined as higher cycles and boundaries using the "Cartier isomorphism" $R_1 \xrightarrow{\sim} Z_1/B_1$.

For $M \in D(R)$ put $\mathrm{Hod}(M) := \underline{\underline{s}}(R_1 \otimes^{L}_{R} M) \in D(k)$

$$t_{\geq i}\, \mathrm{Hod}(M) := \underline{\underline{s}}(0 \longrightarrow (R_1 \otimes^{L}_{R} M)^i \xrightarrow{d} (R_1 \otimes^{L}_{R} M)^{i+1} \longrightarrow \ldots)$$

$$\tau_{\leq i}\, \mathrm{Hod}(M) := \underline{\underline{s}}(\ldots \longrightarrow (R_1 \otimes^{L}_{R} M)^{i-1} \longrightarrow (Z_1 \otimes^{L}_{R} M)^i \longrightarrow 0)\ .$$

Using the Cartier isomorphism we then get 2 distinguished triangles

$$(3.4)\qquad \begin{aligned} &\longrightarrow t_{\geq i+1}\, \mathrm{Hod}(M) \longrightarrow t_{\geq i}\, \mathrm{Hod}(M) \longrightarrow (R_1 \otimes^{L}_{R} M)^i \longrightarrow \\ &\longrightarrow \tau_{\leq i-1}\, \mathrm{Hod}(M) \longrightarrow \tau_{\leq i}\, \mathrm{Hod}(M) \longrightarrow (R_1 \otimes^{L}_{R} M)^i \longrightarrow \end{aligned}$$

Recall (cf. [Il-Ra]) that if M is an R-module then we put

Defin of $\widetilde{\tau}_{\leq}$

$$F^{\infty}BM^i := \bigcup_j F^j dM^{i-1}$$

$$V^{-\infty}ZM^i := \bigcap_j \mathrm{Ker}\ dV^j : M^i \longrightarrow M^{i+1}$$

$$\widetilde{\tau}_{\leq i}M := (\ldots \xrightarrow{d} M^{i-1} \xrightarrow{d} M^i \longrightarrow F^{\infty}BM^{i+1} \longrightarrow 0)$$

$$\widetilde{\tau}_{\geq i+1}M := M/\widetilde{\tau}_{\leq i}M$$

$$\mathrm{dom}^i(M) := M^i/V^{-\infty}ZM^i \longrightarrow F^{\infty}BM^{i+1}\ .$$

Note that then $\tilde{\tau}_{\leq i}M$ is the sub-R-module of M generated by the part of M in degrees $\leq i$.

4. Recall (cf. [Ek 1]) that an R-complex M is said to be coherent if M is complete and $R_n \otimes^L_R M$ is a coherent $W_n k$-complex for all n . Recall also (cf. [Ek 2]) that a complete R-complex M bounded in one of the four directions with $R_1 \otimes^L_R M = 0$ is zero and is bounded if $R_1 \otimes^L_R M$ is.

An R-module M is said to be elementary if M is one of the following types

i) M is in degree zero, finitely generated and free as W-module with F bijective.

ii) M is in degree zero, finitely generated and free as W-module with F and V topologically nilpotent.

iii) M is in degree zero, finitely generated as W-module and killed by p with F bijective.

iv) k $(F = V = d = 0)$

v) $$\underline{\underline{U}}_i := \underset{\text{deg } 0}{\prod_{n\geq 0} kV^n} \xrightarrow{d} \underset{\text{deg } 1}{\prod_{n\geq 0} kdV^{n+i}}$$

where $dV^{-i} = F^i d \quad i \geq 0$.

It is proved in [Ek 2] that $M \in D^-(R)$ is coherent iff $H^i(M)$ is a successive extension of degree shifts of elementary R-modules for all i . It is also clear that a coherent R-module M with $\tilde{\tau}_{\leq 0}M = M$ and $\tilde{\tau}_{\leq -1}M = 0$ is a successive extension of elementary R-modules.

The projection $R \longrightarrow R^0$ is a ring homomorphism and therefore we may consider every (graded) R^0-module as an R-module. An R^0-module M thus considered will be called a Dieudonné module if it is concentrated in degree 0 and is finitely generated as a W-module. It is then clear that if M is

a coherent R-module (i.e. coherent as R-complex) and $M^{-1} = 0$ then $V^{-\infty}ZM^0$ is the largest sub-Dieudonné module of M.

Proposition 4.1: i) *Let* M *and* N *be Dieudonné modules. Then* $\mathrm{Ext}^i_R(M,N)^j = 0$ *unless* $j = 0$ $0 \leq i \leq 3$.

ii) *Let* M *be a Dieudonné module. Then* $\mathrm{Ext}^i_R(\underline{\underline{U}}_k,M)^j = 0$ *unless* $j = 0$ $0 \leq i \leq 2$ *or* $j = 1$ $1 \leq i \leq 3$ *and* $\mathrm{Ext}^i_R(M,\underline{\underline{U}}_k)^j = 0$ *unless* $j = 0,1$ $0 \leq i \leq 3$.

iii) $\mathrm{Ext}^i_R(\underline{\underline{U}}_k,\underline{\underline{U}}_\ell)^j = 0$ *unless* $j = 0$ $0 \leq i \leq 2$; $j = 1$ $0 \leq i \leq 3$ *or* $j = 2$ $1 \leq i \leq 3$.

Proof : Assume first that $k = \bar{k}$. Then every elementary R-module is a successive extension of the following modules :

$$R^0/(F^i - V^j) \qquad (i,j) = 1 \quad i > 0$$

$$W/pW \qquad F = \sigma \qquad d = V = 0$$

$$k$$

$$\underline{\underline{U}}_k$$

In [Ek 2 : III, Cor. 1.5.4] a description of $\mathrm{RHom}_R(N,-)$ for N one of those modules is given, except for $\underline{\underline{U}}_k$, $k \neq 0$ which is left to the reader.

The proposition is now proved by inspection and we get $i \leq 2, i \leq 2$ etc instead of $i \leq 3, i \leq 3$ etc. If $k \neq \bar{k}$ then we have (cf. [Il-Ra]) a functor $(-)^F : D^b_c(R) \longrightarrow D^b(\text{pro-}S_{perf})$ where S_{perf} is the topos of étale sheaves on perfect S-schemes such that if we put $\underline{\mathrm{RHom}}_R(-,-) := (\mathrm{RHom}^!_R(-,-))^F$ then $\mathrm{RHom}_R(-,-) = R\Gamma(S_{perf},\underline{\mathrm{RHom}}_R(-,-))$ and $\underline{\mathrm{RHom}}_R(-,-)$ commutes with change of basefield (cf. [Ek 2]). As $R\Gamma(S_{perf},-)$ has amplitude $[0,1]$ and preserves degrees we conclude by the case $k = \bar{k}$ if we can show that $\underline{\mathrm{Ext}}^i_R(M,N)^j(\bar{k}) = 0$ implies $\underline{\mathrm{Ext}}^i_R(M,N)^j = 0$ for M,N coherent but $\underline{\mathrm{Ext}}^i_R(M,N)^j$ is a pro-quasi-algebraic group (cf. [Il-Ra : IV, 3.11]).

Lemma 4.2. $\mathrm{Hom}_R(\underline{\underline{U}}_k,\underline{\underline{U}}_\ell)=0$ *if* $k>\ell$ *and every non-zero* $\underline{\underline{U}}_k \longrightarrow \underline{\underline{U}}_k$ *is an isomorphism.*

Indeed, let $\varphi:\underline{\underline{U}}_k \longrightarrow \underline{\underline{U}}_\ell$ be non-zero. In degree zero it is then non-zero as $\underline{\underline{U}}_k$ is generated by degree 0 and so injective as is any non-zero mapping $k[[V]] \longrightarrow k[[V]]$. By applying $(-)(\tau)$ $\tau(n)=-\ell\,\delta_{1n}$ we can assume $\ell=0$. Now $\varphi:(\mathrm{Ker}\ d:(\underline{\underline{U}}_k)^0 \longrightarrow (\underline{\underline{U}}_k)^1) \longrightarrow (\mathrm{Ker}\ d:(\underline{\underline{U}}_0)^0 \longrightarrow (\underline{\underline{U}}_0)^1)$ is injective but if $k>0$ then the first term is non-zero and the second zero. When $k=0$ and $\varphi\neq 0$ we see that φ induces an isomorphism on $(\mathrm{Ker}\ Fd)^0$ as both are 1-dimensional and so φ is a surjection as $\underline{\underline{U}}_0$ is generated as $\hat{R}$-module by $(\mathrm{Ker}\ Fd)^0$. Hence φ is an isomorphism in degree 0 and so, in degree 1 as d is an isomorphism.

Recall (cf. [Il-Ra]) that a coherent R-module M is a domino if $M^i=0$ $i\neq 0,1$ $V^{-\infty}ZM^0=0$ and $M^1=F^\infty BM^1$, that $\dim_k M^0/VM^0$ is called the dimension of the domino, that the $\underline{\underline{U}}_k$ are precisely the 1-dimensional dominoes and that every domino is a successive extension of 1-dimensional dominoes.

We will also use the notation $\underline{\underline{U}}_k^j := \underline{\underline{U}}_k[j](-j)$ where $[-]$ means shift in complex degree and $(-)$ shift in module degree.

Furthermore, it is easy to see that $\mathrm{Hom}_W(R_1,K/W)$ considered as an R-module by the action induced by right multiplication on R_1 is isomorphic to $\underline{\underline{U}}_0(1)$. Thus

$$(4.3)\qquad \begin{aligned} \mathrm{RHom}_R(M,\underline{\underline{U}}_0) &= \mathrm{Hom}_W(R_1 \otimes^L_R M, K/W)(-1) \\ \mathrm{RHom}_R(\underline{\underline{U}}_0, D(M)) &= R_1 \otimes^L_R M[2](-1) \qquad M\in D^b_c(R) \end{aligned}$$

by the adjunction formula, duality and (3.3) as $D(\underline{\underline{U}}_0)=\underline{\underline{U}}_0[-2](2)$.

5. Let I be a subset of $\mathbb{Z}$. An R-module M is said to be cut at I if $d:M^i \longrightarrow M^{i+1}$ is zero for all $i\in I$. The category of R-modules cut at I is denoted R-cut-I. A complex in $D(R\text{-cut-}I)$ is said to be coherent if it

is coherent as R-complex.

Proposition 5.1. *The functor* $D^b_c(\text{R-cut-I}) \longrightarrow D^b_c(\text{R-mod})$ *is full and faithful with essential image those complexes whose cohomology is cut at* I .

Proof : Let us first prove that if $M,N \in D^b_c(\text{R-cut-I})$ then $\mathrm{RHom}_{\text{R-cut-I}}(M,N)^0 = \mathrm{RHom}_R(M,N)^0$. By dévissage we may assume that $M,N \in \text{R-cut-I}$. We can write $\mathbb{Z}\setminus I$ as a minimal union of intervals $\mathbb{Z}\setminus I = \cup I_j$. Then $M = \oplus M_j$ and $N = \oplus N_j$ with $M^i_j = N^i_j = 0$ if $i \notin I_j$. It is clear that $\mathrm{RHom}_{\text{R-cut-I}}(M_j,N_j)^0 = \mathrm{RHom}_R(M_j,N_j)^0$ for all j and as $\mathrm{RHom}_{\text{R-cut-I}}(M_j,N_{j'})^0 = 0$ if $j \neq j'$ it will suffice to show that $\mathrm{RHom}_R(M_j,N_{j'})^0 = 0$. This follows by dévissage from (4.1).
Now suppose that $M \in D^b_c(R)$ has cohomology which is cut at I . We want to show that there is a complex M' cut at I with M' isomorphic to M , M' is then necessarily unique up to isomorphism. We will do this by induction over the length $n-m$ of the amplitude $[m,n]$ of M . If the length is 0 everything is clear. If not choose $i \in [m,n-1]$. By assymption there are M_1 and M_2 cut at I isomorphic to $\tau_{\leq i}M$ resp. $\tau_{\geq i+1}M$ and so we get a distinguished triangle $M_1 \longrightarrow M \longrightarrow M_2 \xrightarrow{\varphi} M_1[1]$. By fullness there is a morphism $M_2 \longrightarrow M_1[1]$ in $D^b_c(\text{R-cut-I})$ which equals φ in $D^b_c(R)$. We can then let M' be a cone of this $M_2 \longrightarrow M_1[1]$ in $D^b_c(\text{R-cut-I})$.

Remark : We will not particularly distinguish between $D^b_c(\text{R-cut-I})$ and its essential image in $D^b_c(R)$ and hence we will say that a complex whose cohomology is cut at I is cut at I .

If I is an intervall of $\mathbb{Z}$ then we denote by R-mod-I the category of R-modules of level I . The proof of the following result is then altogether similar to the proof of (5.1).

Proposition 5.2. *The functor* $D^b_c(\text{R-mod-I}) \longrightarrow D^b_c(R)$ *is full and faithful with essential image those complexes whose cohomology is of level* I .

<u>Remark</u> : The same sort of abuse of language will be used here.

Recall also that if $M \in D(R)$ then considering M as a double complex we get the slope spectral sequence :

(5.3) $$E_1^{i,j} = H^j(M)^i \Longrightarrow H^{i+j}(\underline{\underline{s}}(M)) .$$

6. Let $M \in D^b_c(R)$. Then we put

(6.1)
$$\begin{aligned}
h^{i,j}(M) &:= \dim H^j(R_1 \otimes^L_R M)^i \\
T^{i,j}(M) &:= \dim H^j(M)^i/(V^{-\infty}ZH^j(M)^i+V) \\
m^{i,j}(M) &:= \dim H^j(M)^i/(p\text{-torsion}+V)+\dim H^{j+1}(M)^{i-1}/(p\text{-tors}+F) \\
b_n(M) &:= \dim_K H^n(\underline{\underline{s}}(M)) \otimes K
\end{aligned}$$

where $K := \mathrm{Frac}(W) = W \otimes_{\mathbb{Z}} \mathbb{Q}$.

Then

<u>Proposition</u> 6.2. i) $b_n(M) = \sum_{i+j=n} m^{i,j}(M) \leq \sum_{i+j=n} h^{i,j}(M)$.

ii) *If* $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$ *are the slopes of the iso*-F-*crystal* $H^n(\underline{\underline{s}}(M)) \otimes K$ *then*

$$m^{i,n-i}(M) = \sum_{\lambda_s \in [i,i+1[} (i+1-\lambda_s) + \sum_{\lambda_s \in [i-1,i[} (\lambda_s-i+1) .$$

Indeed, $\sum_{i+j=n} m^{i,j}(M) = \sum_{i+j=n} \dim H^j(M)^i/(p\text{-tors}+V) +$

$\dim H^j(M)^i/(p\text{-tors}+F) = \sum_{i+j=n} \dim H^j(M)^i/(p\text{-tors}+p) = \sum_{i+j=n} \dim H^j(M)^i \otimes K =$

$\dim H^n(\underline{\underline{s}}(M)) \otimes K$ the last as (5.3) degenerates modulo torsion (cf. [Ek 2] or (I : Prop. 2.3)).

Also (3.1) and (1.5) give us a spectral sequence

$$(6.2.1) \qquad E_1^{j,i} = H^i(R_1 \otimes_R^L M)^j \Longrightarrow H^{i+j}(k \otimes_W^L \underline{\underline{s}}(M))$$

and so $\sum_{i+j=n} h^{i,j}(M) \geq \dim H^n(k \otimes_W^L \underline{\underline{s}}(M)) \geq \dim H^n(\underline{\underline{s}}(M))/(p\text{-tors}+p) = b_n(M)$.
As for ii) it is simply the formula for $\dim M/V$ and $\dim M/F$ of a torsion free Dieudonné module in terms of its slopes ; it is proven by reduction to the case of one single slope and computation.

7. We will throughout this paper carry the action of a finite group H along. In most of our results and proofs this action follows along automatically, sometimes we reduce to $H = 1$. This will not always be mentioned. The essential formula is

$$\mathrm{RHom}_{R[H]}(M[H],N) = \mathrm{RHom}_R(M,N) \quad \text{for}$$

$$M \in D(R) \quad \text{and} \quad N \in D^+(R[H]) .$$

8. The following result is proved by explicit computation using [Ek 2 :0:5.4].

<u>Lemma</u> 8.1. *If* M *is an elementary* R*-module and* D *is one of* R_n , Z_n , B_n , $\sigma_*^n R_1/B_n$ *then* $H^i(D \otimes_R^L M)^j = 0$ *unless* $i = 0$, $j = 0,1$; $i = 1$, $j = 0,1,2$ *or* $i = 2$ $j = 1,2,3$.

I

The diagonal t-structure

Theorem 1.1. *The functorial filtration* $\cdots \subseteq \tilde{\tau}_{\leq i} \subseteq \tilde{\tau}_{\leq i+1} \subseteq \cdots$ *is radical on* $D_c^b(R)$ *with the standard t-structure.*

By (0:1.4i) we need to show that if M and N are coherent R-modules then $\mathrm{Hom}^r_{D_c^b(R)}(\tau_{\leq s}(M)/\tau_{\leq s-1}(M), \tau_{\leq t}(N)/\tau_{\leq t-1}(N)) = 0$ when $r < t-s$.

By degree shifting we may assume that $s = 0$.

By dévissage we may assume that M and N are elementary and $H = 1$ in which case we conclude by (0:4.1).

Definition 1.2. i) *The t-structure associated to the radical filtration of* Thm. 1.1 *will be denoted* $(\tilde{D}_c^b(R)^{\leq 0}, \tilde{D}_c^b(R)^{\geq 0})$, *the truncation functors* $\tilde{\tau}_{\leq i}$ *and* $\tilde{\tau}_{\geq i}$ *and* $\Delta := \tilde{D}_c^b(R)^{\leq 0} \cap \tilde{D}_c^b(R)^{\geq -1}$, *its objects will be called diagonal complexes. Furthermore we will put* $\tilde{H}^i(-) := \tilde{\tau}_{\leq i} \circ \tilde{\tau}_{\geq i}(-)[i] : D_c^b(R) \longrightarrow \Delta$.

It is clear that the diagonal t-structure is compatible with change of base field and change of group. What happens during change of groups is easy to understand in view of the following proposition.

Proposition 1.3. *The category of diagonal* H*-complexes is equivalent to the category of ordinary diagonal complexes with given* H*-actions.*

Proof : This follows from [B-B-D:Thm. 3.24] applied to the topos of H-sets and the covering $H \longrightarrow 1$ of the final object.

Note that the diagonal t-structure is stable under the functor $[i](-i)$. We will refer to any of those functors as shifting. Hence the filtration $\ldots \subseteq \tau_{\leq i} \subseteq \tau_{\leq i+1} \subseteq \ldots$ of Δ have quotients which are shifts of elements in $\Delta \cap$ R-mod . A coherent R-module M is clearly in Δ iff $M^i = 0$ if $i \neq 0,1$ and $F^{\infty}BM^1 = M^1$ and hence exactly the extensions of Dieudonné modules by dominoes.

Proposition 1.4. *Let* $M \in \Delta$

i) $H^i(\underline{\underline{s}}(M)) = 0$ *if* $i \neq 0,1$ *and* $H^1(\underline{\underline{s}}(M))$ *is torsion*

ii) $H^i(D \otimes_R M)^j = 0$ *if* $i+j \neq -1,0,1$ *and* D *is any of the right* R-*modules* R_n , B_n , Z_n , $\sigma_*^n R_1/B_n$.

Proof : By dévissage and shifting we may assume that M is an elementary R-module and then we apply (0:8.1) for ii) whereas i) is obvious.

Corollary 1.4.1. *Let* $M \in D_c^b(R)$.

i) *There are short exact sequences*

$$0 \longrightarrow H^1(\underline{\underline{s}}(\widetilde{H}^{i-1}(M))) \longrightarrow H^i(\underline{\underline{s}}(M)) \longrightarrow H^0(\underline{\underline{s}}(\widetilde{H}^i(M))) \longrightarrow 0$$

ii) *If* D *is as above then* $H^i(D \otimes_R^L \widetilde{H}^j(M))^{-i}$ *is a subquotient of* $H^{i+j}(D \otimes_R^L M)^{-i}$.

This follows from the spectral sequence (0:1.5) applied to $H^0(\underline{\underline{s}}(-))$ and $H^0(D \otimes_R^L (-))^{-i}$ respectively plus the vanishing of i) resp. ii).

It is clear that $\underline{\underline{U}}_0$ is neither noetherian nor artinian as a coherent R-module. The following result therefore comes as a surprise.

Proposition 1.5. *The category* Δ *is noetherian.*

Lemma 1.5.1. *Let* $M \in \Delta \cap R\text{-mod}$.

i) *If* $N \hookrightarrow M$ *is a subobject in* Δ *then* $N \in \Delta \cap R\text{-mod}$ *and* $N^0 \longrightarrow M^0$ *is a monomorphism of* R-*modules.*

ii) *If* $N \in R\text{-mod} \cap \Delta$ *and* $N \longrightarrow M$ *is a morphism s.t.* $N^0 \longrightarrow M^0$ *is a monomorphism of* R-*modules then* $N \longrightarrow M$ *is a monomorphism.*

Proof : Let M/N be a quotient in Δ . We then get a distinguished triangle $\longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow N[I] \longrightarrow$ and a long exact sequence of R-modules
$\ldots \longrightarrow H^{-1}(M) \longrightarrow H^{-1}(M/N) \longrightarrow H^0(N) \longrightarrow H^0(M) \longrightarrow H^0(M/N) \longrightarrow \ldots$
I claim that $H^i(N)^{-i} \longrightarrow H^i(M)^{-i}$ is injective for all i . This follows as $H^{i-1}(M/N)^{-i} = 0$. Hence $H^i(N)^{-i} = 0$ if $i \neq 0$ but $H^i(N)$ is generated by $H^i(N)^{-i}$ so $H^i(N) = 0$ and $N \in \Delta \cap R\text{-mod}$ and $N^0 \longrightarrow M^0$ is injective. Conversely if $N \in \Delta \cap R\text{-mod}$ and we have $\varphi : N \longrightarrow M$ with $N^0 \longrightarrow M^0$ injective then $\operatorname{Ker} \varphi \in R\text{-mod} \cap \Delta$ and $(\operatorname{Ker} \varphi)^0 \longrightarrow N^0$ is injective by i) but as $N^0 \longrightarrow M^0$ is injective $(\operatorname{Ker} \varphi)^0 \hookrightarrow 0$ and $\operatorname{Ker} \varphi = 0$ being generated by $(\operatorname{Ker} \varphi)^0$.

To show that Δ is noetherian we may assume $H = 1$ by (1.3) and by dévissage and shift that M is in $R\text{-mod} \cap \Delta$ and that M is either a Dieudonné module which is taken care of by (1.5.1 i) or that $M = \underline{\underline{U}}_j$ for some j . Let $N_i \subseteq N_{i+1} \subseteq \ldots \subseteq \underline{\underline{U}}_j$ be a sequence of subobjects. By (1.5.1 i) all the N_i are 1-dimensional dominoes hence $\simeq \underline{\underline{U}}_{r_i}$ for some r_i . By (0:4.2) $r_i \leq r_{i+1} \leq \ldots \leq j$ and hence there is some i_0 s.t. $r_i = r_{i_0}$ $i \geq i_0$. Again by (0:4.2) $N_i = N_{i_0}$ $i \geq i_0$.

To investigate the artinian properties of Δ we will use duality. Before we will be able to do this we need the following characterization of the Δ-amplitude.

Proposition 1.6. i) $(\tilde{D}^b_c(R)^{\leq 0}, \tilde{D}^b_c(R)^{\geq 0})$ *is stable under* $(-)(\varphi)$ *for all* $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$.

ii) $M \in D^b_c(R)$ *has* Δ*-amplitude* $[m,n]$ *iff there is an* $N \in \mathbb{Z}$ *such that if* $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ *has* $\varphi(n) \geq N$ *for all* n *then* $\underline{\underline{s}}(M(\varphi))$ *has amplitude* $[m,n]$.

<u>Proof</u> : i) is clear.

Suppose we can prove that for $M \in \Delta$

a) if $H^0(\underline{\underline{s}}(M(\varphi))) = 0$ is 0 for all φ with $\varphi(n) \geq N$ for some N then $N = 0$.

b) There exists an N such that $H^1(\underline{\underline{s}}(M(\varphi))) = 0$ for all φ with $\varphi(n) \geq N$.

Then ii) follows from (1.4.1 i) and (1.4 i) in view of i) which tells us that $\widetilde{H}^i(M(\ )) = \widetilde{H}^i(M)(\varphi)$.

As for a) if it is false we may assume that $H^r(M)$ is 0 for $r < r_0$ and non-zero for $r = r_0$ for some r_0 . The spectral sequence

$$E_2 = H^i(\underline{\underline{s}}(H^r(M(\varphi)))) \Longrightarrow H^{i+r}(\underline{\underline{s}}(M(\varphi)))$$

shows that $H^0(\underline{\underline{s}}(H^r(M(\varphi))(r))) = 0$ so we may assume that $M \in \Delta \cap R\text{-mod}$. Then we see immediately that $V^{-\infty}ZM^0 = 0$ so that M is a domino. By just forgetting the H-action we may assume that $H = 1$. Then M contains a $\underline{\underline{U}}_i$ for some i but then $H^0(\underline{\underline{s}}(\underline{\underline{U}}_i(\varphi))) \hookrightarrow H^0(\underline{\underline{s}}(M(\varphi)))$ and $H^0(\underline{\underline{s}}(\underline{\underline{U}}_i(\varphi))) \neq 0$ if $\varphi(n) \geq -i+1$ for all n . Now for b) by dévissage and shifting we may assume that $M \in \Delta \cap R\text{-mod}$ and $F^N d : M^0 \longrightarrow M^1$ is surjective for some N and this N will do.

Now it follows from [Ek 1 : IV, 4.4] that $D((-)(\varphi)) = D(-)(-\varphi-)$ where $(-\varphi-)(n) = -\varphi(-n)$. From this, (1.6) and the formula $\underline{\underline{s}}(D(-)) = R\mathrm{Hom}_W(\underline{\underline{s}}(-),W)$ we see that $D(-)$ has Δ-amplitude $[0,1]$. We put $\widetilde{D}^i(-) := \widetilde{H}^i \circ D(-)$ and hence $\widetilde{D}^i(-) = 0$ on Δ if $i \neq 0,1$. By biduality, $D(D(-)) = \mathrm{id}$, we get a spectral sequence in Δ for any $M \in \Delta$

$$E_2 = \widetilde{D}^i(\widetilde{D}^j(M)) \Longrightarrow M \ . \tag{1.7}$$

This plus our vanishing gives

(1.8) $$\tilde{D}^0(\tilde{D}^1(M)) = \tilde{D}^1(\tilde{D}^0(M)) = 0$$

and a short exact sequence, natural in M ,

(1.9) $$0 \longrightarrow \tilde{D}^1(\tilde{D}^1(M)) \longrightarrow M \longrightarrow \tilde{D}^0(\tilde{D}^0(M)) \longrightarrow 0 .$$

We put $t^0(-) := \tilde{D}^0(\tilde{D}^0(-))$ and $t^1(-) := \tilde{D}^1(\tilde{D}^1(-))$.

Definition 1.10. *Let* $M \in \Delta$

i) M *is finite torsion if* $t^1(M) = M$.

ii) M *is without finite torsion if* $t^1(M) = 0$.

Proposition 1.11. *Let* $M \in \Delta$

i) M *is without finite torsion iff* $t^0(M) = M$.

ii) $\tilde{D}^1(M)$ *is finite torsion and* $\tilde{D}^0(M)$ *without finite torsion.*

iii) $t^0(M)$ *is the maximal quotient of* M *without finite torsion and is the maximal finite torsion subobject.*

iv) *A sequence* $\ldots \subseteq M_{i+1} \subseteq M_i \subseteq \ldots \subseteq M$ *of subobjects of* M *such that* M/M_i *is without finite torsion is constant for all* i *sufficiently large.*

Indeed, (1.8) and (1.9) shows that $t^1(t^1(M)) = t^1(M)$ and $t^1(M/t^1(M)) = t^1(t^0(M)) = 0$ so t^1 is an idempotent radical and i) and iii) follows whereas ii) follows directly from (1.8). Let us apply $\tilde{D}^0(-)$ to $M \ldots \longrightarrow M/M_i \longrightarrow M/M_{i+1} \longrightarrow \ldots$ This gives us an ascending sequence $\ldots \subseteq \tilde{D}^0(M/M_i) \subseteq \tilde{D}^0(M/M_{i+1}) \subseteq \ldots$ of subobjects of $\tilde{D}^0(M)$. Hence there is some is such that $\tilde{D}^0(M/M_{i_0}) = \tilde{D}^0(M/M_i)$ for $i \geq i_0$ and by applying once again $\tilde{D}^0(-)$ we get $t^0(M/M_{i_0}) = t^0(M/M_i)$ $i \geq i_0$ but by assumption $M/M_{i_0} = t^0(M/M_{i_0})$ and $M/M_i = t^0(M/M_i)$.

<u>Theorem</u> 1.12. *Let* $M \in \Delta$

i) M *is finite torsion iff* $H^i(M)(i)$ *is a Dieudonné module of finite length for all* i *and then* $M = \bigoplus_i H^i(M)[-i]$.

ii) M *is without finite torsion iff* $H^i(M)(i)$ *is without finite torsion for all* i *and if* $M \in R\text{-mod} \cap \Delta$ *it is without finite torsion iff* $V^{-\infty}ZM^0$ *is a torsion free* W*-module.*

<u>Proof</u> : If we can prove that when M is finite torsion and $H^r(M) = 0$ for $r > r_0$ for some r_0 then $H^{r_0}(M)(r_0)$ is a Dieudonné module of finite length then we have proved the only if-part of i). This is because then M would be cut at $\{r_0\}$ so $M = \tau_{\leq r_0 - 1}M \oplus H^{r_0}(M)[-r_0]$ and a direct factor of something which is finite torsion is of course finite torsion so we could proceed by induction. However, $H^{r_0}(M)[-r_0]$ being a quotient of M is finite torsion so we are reduced to proving i) for $H^{r_0}(M)[-r_0]$ and by shifting we may assume $M \in \Delta \cap R\text{-mod}$. We may clearly assume $H = 1$. If M is not a Dieudonné module it has a quotient isomorphic to $\underline{\underline{U}}_i$, for some i , which would then be finite torsion. However, if $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ has $\varphi(0) < -i$ then $H^1(\underline{\underline{s}}(\underline{\underline{U}}_i(\varphi))) \neq 0$ so $H^0(\underline{\underline{s}}(D(\underline{\underline{U}}_i)(-\varphi-)) \neq 0$ which by (1.4 i) and (1.4.1 i) contradicts the assumption that $D(\underline{\underline{U}}_i) = \widetilde{D}^1(\underline{\underline{U}}_i)[-1]$. Hence M is a Dieudonné module. If M is not of finite length then M has a quotient which is a torsion free Dieudonné module. Then $H^0(\underline{\underline{s}}(D(M)) = \mathrm{Hom}_W(M,W)$ would be non-zero again contradicting $D(M) = \widetilde{D}^1(M)[-1]$. Conversely if M is a Dieudonné module of finite length then for every $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ $M(\varphi) = M$ so $\underline{\underline{s}}(D(M)(\varphi)) = R\mathrm{Hom}_W(M,W)$ so (1.6) shows that $D(M)$ is of amplitude [1,1] i.e. equal to $\widetilde{D}^1(M)[-1]$ and by (1.8) $M = D(D(M)) = \widetilde{D}^1(\widetilde{D}^1(M)) = t^1(M)$. By dévissage and shift we get the if-part. Now for ii) we may assume that $H^r(M) = 0$ for $r > r_0$ and as $\tau_{\leq r_0 - 1}$ is a subobject of M and hence without finite torsion we can by induction assume that $H^r(M)(r)$ is without finite forsion for $r < r_0$. Hence it remains to show that $H^{r_0}(M)(r_0)$ is without

finite torsion. By shifting we may assume that $r_0 = 0$. Let us consider the exact sequence in Δ

$$0 \longrightarrow \tau_{\leq -1}M \longrightarrow M \longrightarrow H^0(M) \longrightarrow 0 .$$

We get an exact sequence

$$\widetilde{D}^0(M) \xrightarrow{\varphi} \widetilde{D}^0(\tau_{\leq -1}(M)) \longrightarrow \widetilde{D}^1(H^0(M)) \longrightarrow 0$$

because $\widetilde{D}^1(M) = 0$. Put $T := \mathrm{Im}(\varphi)$ and $N := \widetilde{D}^0(T)$. Then $N \hookrightarrow \widetilde{D}^0(\widetilde{D}^0(M)) = M$ and we get an exact sequence $0 \longrightarrow \tau_{\leq -1}(M) = D^0D^0(\tau_{\leq -1}(M)) \longrightarrow N \longrightarrow$ $t^1(H^0(M)) \longrightarrow 0$ because of (1.8). Now as $N \hookrightarrow M$ N is without finite torsion. On the other hand as $H^0(\tau_{\leq -1}(M)) = H^1(\tau_{\leq -1}(M)) = 0$ $H^0(N) = H^0(t^1(H^0(M)))$ so by i) N is cut at 0 so by (0:5.1) $\tau_{\geq 0}t^1(H^0(M))$ is a direct factor of N so it is at the same time finite torsion and without finite torsion and hence zero. Finally $t^1(H^0(M))$ is a subobject of $H^0(M) \in R\text{-mod} \cap \Delta$ and by Lemma 1.5.1 this implies that $t^1(H^0(M)) \in R\text{-mod} \cap \Delta$ and so $0 = \tau_{\geq 0}t^1(H^0(M)) = t^1(H^0(M))$. Let $M \in \Delta \cap R\text{-mod}$. By Lemma 1.5 any subobject of M in Δ is in $R\text{-mod} \cap \Delta$ and if it is a Dieudonné module it is a subobject also in R-mod. By (1.5.1 ii) if $N \in R\text{-mod} \cap \Delta$ is a Dieudonné module and $N \longrightarrow M$ is a monomorphism in R-mod then it is also a monomorphism in Δ . Hence by i) M is without torsion iff it does not contain a sub-R-module N which also lies in Δ and is finite torsion as such and those N are exactly the Dieudonné modules of finite length so M is without finite torsion iff $V^{-\infty}ZM^0$ is torsion free it containing all sub-Dieudonné modules of M . The only if-part of ii) is clear as extensions of objects without finite torsion are without finite torsion.

Hodge-Witt objects

2. We begin by showing that the diagonal Hodge-Witt objects form a nice category. Let Δ_{HW} denote the category of diagonal complexes which also

are Hodge-Witt complexes, i.e. cut at $\mathbb{Z}$.

Proposition 2.1. Δ_{HW} *is closed under sub and quotient objects and extensions.*

Indeed, extensions are clear as the Hodge-Witt complexes form a triangulated category. Let $M \in \Delta_{HW}$ and N a subobject in Δ . As was seen in the proof of Lemma 1.5.1 $H^i(N)^{-i} \hookrightarrow H^i(M)^{-i}$ and so $H^i(N)^{-i}$ is a finitely generated W-module which implies that $V^{-\infty}ZH^i(N)^{-i} = H^i(N)^{-i}$ and $H^i(N)(i)$ is a Dieudonné module. Again the long exact sequence of Lemma 1.5.1 shows that now $H^i(M)^{-i} \longrightarrow H^i(M/N)^{-i}$ is surjective so the same argument shows that $H^i(M/N)^{-i}$ is a Dieudonné module.

Proposition 2.2. *Let* $M \in \Delta$

i) ${}_pM \in \Delta_{HW}$ *iff* ${}_pM := \mathrm{Ker}\; p : M \longrightarrow M \in \Delta_{HW}$.

ii) $M \in \Delta_{HW}$ *iff there exists a constant* C *such that* $\mathrm{lgth}(H^1(\underline{s}(M(\varphi)))) < C$ *for all* $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$.

Indeed, only if of i) follows from (2.1). Suppose that ${}_pM \in \Delta_{HW}$ and choose r_0 such that $H^r(M) = 0$ if $r > r_0$. Then $\tau_{\leq r_0 - 1}M$ is a subobject of M so ${}_p(\tau_{\leq r_0 - 1}M)$ is Hodge-Witt by (2.1) and by induction we may assume that $\tau_{\leq r_0-1}M$ is Hodge-Witt. Snake lemma applied to multiplication by p on the exact sequence $0 \longrightarrow \tau_{\leq r_0-1}M \longrightarrow M \longrightarrow H^{r_0}(M)[-r_0] \longrightarrow 0$ and (2.1) shows that $pH^{r_0}(M)[-r_0]$ is Hodge-Witt so by shifting we may assume that $M \in R\text{-mod} \cap \Delta$. Lemma 1.5.1 shows that ${}_pM \in R\text{-mod} \cap \Delta$ and that ${}_p({}_pM)^0 = {}_p(M^0)$ so we are reduced to showing that ${}_p(M^0)$ of finite length implies that $V^{-\infty}ZM^0 = M^0$. Look however at the exact sequence

$$0 \longrightarrow V^{-\infty}ZM^0 \longrightarrow M^0 \longrightarrow M^0/V^{-\infty}ZM^0 \longrightarrow 0$$

and apply snake lemma for multiplication by p. As $V^{-\infty}ZM^0/p$ is of finite length we see that ${}_pM^0/V^{-\infty}ZM^0$ is of finite length but if $M^0/V^{-\infty}ZM^0$ is non-zero it contains $k[[V]]$ as a submodule. For ii) the only if is clear as $M(\varphi) = M$ if $M \in \Delta_{HW}$. In the other direction we may suppose again that $H^r(M) = 0$ $r > r_0$. If we can show that $H^{r_0}(M)[-r_0]$ is Hodge-Witt then $M = \tau_{\leq r_0 - 1}M \oplus H^{r_0}(M)[-r_0]$ and we can continue by induction. Now the spectral sequence

$$H^i(\underline{\underline{s}}(H^j(M(\varphi))) \Longrightarrow H^{i+j}(\underline{\underline{s}}(M(\varphi)))$$

shows that $H^1(\underline{\underline{s}}((H^{r_0}(M)[-r_0])(\varphi)))$ is a quotient of $H^1(\underline{\underline{s}}(M(\varphi)))$ so our conditions are fulfilled for $H^{r_0}(M)[-r_0]$ and by shifting we may assume that $M \in R\text{-mod} \cap \Delta$ but if M then is not Hodge-Witt it has a quotient isomorphic to $\underline{\underline{U}}_i$ so $H^1(\underline{\underline{s}}(\underline{\underline{U}}_i(\varphi)))$ would be bounded which is clearly false.

<u>Corollary</u> 2.2.1. *An* $M \in D_c^b(R)$ *is Hodge-Witt iff the length of the torsion of* $H^*(\underline{\underline{s}}(M(\varphi)))$ *is bounded independently of* φ.

This follows from Cor. 1.4.1.

By (2.1) and (0:1.1) we see that for any $M \in \Delta$ there is a largest Hodge-Witt subobject which we will denote $HW(M)$. Then $HW(-)$ is an idempotent radical.

We then have

<u>Proposition</u> 2.2.2. $HW(H^i(-)) = H^i(HW(-))$.

<u>Proof</u> : It will be enough to show that if $M \in \Delta$ then $H^i(HW(M)) \longrightarrow H^i(M)$ is injective and that if $HW(M) = 0$ then $HW(H^i(M)) = 0$. First $H^i(HW(M))[-i] \hookrightarrow M$ with quotient N. Then $H^{i-1}(N) \longrightarrow H^i(H^i(HW(M))[-i]) = H^i(HW(M))$ is zero as $H^{i-1}(N)$ is generated in degree $i-1$ and $H^i(HW(M))$ is concentrated in degree i. By the long exact cohomology sequence $H^i(HW(M)) \longrightarrow H^i(M)$ is injective.

Let now $HW(M) = 0$. Now $HW(H^i(M))[-i] \hookrightarrow \tau_{\geq i}M$. Let N be the inverse in M of $HW(H^i(M))[-i]$. Then N is cut at $\{i\}$ so $HW(H^i(M))[-i]$ is a direct factor of N and so a subobject of M and hence 0 .

With the results obtained so far we can give a new proof of "survie du coeur" as well as a generalization of [Il 1].

Proposition 2.3. *Let* $M \in D_c^b(R)$

i) *(survie du coeur) In the spectral sequence*

$$E_1^{j,i} = H^i(M)^j \Longrightarrow H^{i+j}(\underline{s}(M))$$

$$B_\infty^{i,j} \subseteq F^\infty BH^j(M)^i \subseteq V^{-\infty}ZH^j(M)^i \subseteq Z_\infty^{i,j} .$$

ii) *In the same spectral sequence the torsion of* $\mathrm{Fil}^j H^{i+j}(\underline{s}(M))$ *maps onto the torsion in* $V^{-\infty}ZH^j(M)^i/B_\infty^{i,j}$.

Indeed, we have a morphism $\tilde{\tau}_{i+j-1}M \longrightarrow M$ which induces an isomorphism $H^k(\tilde{\tau}_{\leq i+j-1}M)^\ell \longrightarrow H^k(M)^\ell$ if $k+\ell < i+j$ and an injection if $k+\ell = i+j$ whose image, when $k+\ell = i+j$ is $F^\infty BH^k(M)^\ell$. The inclusion $B_2^{i,j} \subseteq F^\infty BH^j(M)^i$ now follows by functoriality of the spectral sequence. That $B_r^{i,j} \subseteq F^\infty BH^j(M)^i$ follows from repeated application of the following useful lemma which shows that $E_r^{k,\ell}(\tau_{\leq i+j-1}M) = E_r^{k,\ell}(M)$ if $k+\ell < i+j$ and $E_r^{k,\ell}(\tau_{\leq i+j-1}M) \subseteq E_r^{k,\ell}(M)$ if $k+\ell = i+j$.

Lemma 2.3.1. *If* $f : X^\cdot \longrightarrow Y^\cdot$ *is a morphism of complexes in an abelian category and if* f^i *is surjective and* f^{i+1} *injective then so is* $H(f)^i$ *resp.* $H(f)^{i+1}$.

Proof : Regard $X^\cdot \longrightarrow Y^\cdot$ as a double complex and consider the two spectral sequences thus obtained or chase diagrams.

For the other inclusion consider the t-structure $\hat{D}$ associated to the idempotent radical $HW(-)$ on Δ. Then $\hat{\tau}_{\leq i+j-1}M \longrightarrow M$ induces an isomorphism

$$V^{-\infty}ZH^j(\hat{\tau}_{i+j-1}M)^i \longrightarrow V^{-\infty}ZH^j(M)^i \quad \text{and}$$

$H^k(\hat{\tau}_{i+j-1}M)^\ell = 0$ if $k+\ell > i+j$.

As for ii), $F^\infty BH^i(M)^j/B_\infty^{i,j}$ is taken care of by (1.4 i) and (1.4.1 i) by considering $\tilde{\tau}_{\leq i+j-1}M \longrightarrow M$. For the rest we reduce using (1.4 i), (1.4.1 i) and i) to $\tilde{\tau}_{\geq i+j}M$ and then trivially to $\tilde{H}^{i+j}(M)$ so we may assume that $M \in \Delta$. Theorem 1.2.2 shows that $t^1(M) \longrightarrow M$ is onto on the torsion of $H^{i+j}(-)$ so we reduce to $t^1(M)$ but it is Hodge-Witt so there is nothing to prove.

We are now almost prepared to give the local version of the Hodge-Witt decomposition. What remains is only to give a generalization of the notion of exotic torsion (cf. [Il 1]).

<u>Definition</u> 2.4. *Let* $M \in D_c^b(R)$.

i) *The exotic* 1-*torsion in degree* i *of* M *is the image of* $H^1(\underline{\underline{s}}(\tilde{H}^{i-1}(M)))$ *in* $H^i(\underline{\underline{s}}(M))$.

ii) *The finite total torsion in degree* i *of* M *is* $H^0(\underline{\underline{s}}(t^0(\tilde{H}^i(M))))$.

iii) *The exotic* 0-*torsion in degree* i *of* M *is the torsion of* $H^0(\underline{\underline{s}}(t^1(\tilde{H}^i(M))))$.

In this way we get a 3-step filtration of the torsion of $H^i(\underline{\underline{s}}(M))$ whose quotients are respectively the exotic 1-torsion, the finite total torsion and the exotic 0-torsion. If $N = D(M)$ then the formula $\underline{\underline{s}}(D(-)) = RHom_W(\underline{\underline{s}}(-),W)$ shows that the torsion in degree i of $\underline{\underline{s}}(M)$ is dual to the torsion in degree $-i+1$ of $\underline{\underline{s}}(N)$ and it is clear that the filtrations just described are dual to each other. Finally, if X is a smooth

and proper variety then the divisorial and exotic torsion of $H^2_{cris}(X/W)$ in the sense of [Il 1:II 6.7] equals the total finite torsion and the exotic 0-torsion respectively.

<u>Theorem</u> 2.5. *Let* $M \in D^b_c(R)$ *and suppose that* $d : H^i(M)^j \longrightarrow H^i(M)^{j+1}$ *is zero. Then*

i) *The quotient of* $H^{i+j}(\underline{\underline{s}}(M))$ *by its exotic* 1*-torsion admits a canonical decomposition* $H^{\leq i} \oplus H^{>i}$.

ii) *The differentials* $d_r : E_r^{k,\ell} \longrightarrow E_r^{k+r,\ell-r+1}$ *of the spectral sequence* $H^i(M)^j \Longrightarrow H^{i+j}(\underline{\underline{s}}(M))$ *are zero when* $k+\ell = i+j$ $i-r < k \leq i$.

The proof of (2.3) allows us to reduce to $\widetilde{H}^{i+j}(M)[-i-j]$ but by assumption this complex is cut at $\{j\}$ so we can apply (0:5.1).

We say that an $M \in D^b_c(R)$ with $d : H^i(M)^j \longrightarrow H^i(M)^{j+1}$ zero whenever $i+j = r$ is Hodge-Witt in degree r . We then get a Hodge-Witt decomposition for $H^r(\underline{\underline{s}}(M))$ modulo exotic 1-torsion for all M which are Hodge-Witt in degree r .

<u>Remark</u> : This definition of Hodge-Witt in degree r differs from the one used by Illusie and Raynaud in [Il-Ra]. Their "Hodge-Witt in degree r" is equivalent to our "Hodge-Witt in degree r and $r-1$". I suggest the present definition because it has the desirable feature that M is Hodge-Witt in degree r iff $\widetilde{H}^r(M)$ is Hodge-Witt. From our point of view the only role played by the assumption "H-W in degree r-1" to insure a H-W decomposition in degree r is that it excludes the existence of exotic 1-torsion in degree r . Many other assumptions will give us that.

<u>The two fundamental filtrations</u>

3. One important invariant of a diagonal complex is the associated simple complex. We start therefore by dividing Δ in classes according to the behaviour of $\underline{\underline{s}}(-)$.

Definition 3.1. *An* $M \in \Delta$ *is said to be*

i) *without* $\underline{\underline{s}}$*-torsion if* $H^*(\underline{\underline{s}}(M))$ *is without torsion*

ii) $\underline{\underline{s}}$*-torsion if* $H^*(\underline{\underline{s}}(M))$ *is torsion.*

iii) $\underline{\underline{s}}$*-0-torsion* ($\underline{\underline{s}}$*-1-torsion) if it is* $\underline{\underline{s}}$*-torsion and* $H^1(\underline{\underline{s}}(M)) = 0$ $(H^0(\underline{\underline{s}}(M)) = 0)$.

iv) $\underline{\underline{s}}$*-acyclic if* $\underline{\underline{s}}(M) = 0$.

The following lemma is then crucial

Lemma 4.2 i) *The* $\underline{\underline{s}}$*-torsion objects are closed under sub and quotient objects and extensions.*

ii) *The* $\underline{\underline{s}}$*-1-torsion* ($\underline{\underline{s}}$*-0-torsion) objects are closed under extensions and sub (quotient) objects.*

iii) $\underline{\underline{s}}$*-1-torsion objects are without finite torsion.*

iv) *An element* $M \in D^b_c(R)$ *has* $\underline{\underline{s}}(M) = 0$ *iff* $\widetilde{H}^i(M)$ *is* $\underline{\underline{s}}$*-acyclic for all* i .

Proof : Proposition 1.4 shows that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in Δ then we get a long exact sequence

$$(4.2.1) \quad 0 \to H^0(\underline{\underline{s}}(M_1)) \to H^0(\underline{\underline{s}}(M_2)) \to H^0(\underline{\underline{s}}(M_3)) \to H^1(\underline{\underline{s}}(M_1)) \to H^1(\underline{\underline{s}}(M_2)) \to H^1(\underline{\underline{s}}(M_3)) \to 0$$

and $H^1(\underline{\underline{s}}(M))$ is torsion for all $M \in \Delta$. From this i) and ii) follows immediately. Let M be $\underline{\underline{s}}$-1-torsion. By ii) $t^1(M)$ is $\underline{\underline{s}}$-1-torsion but by (1.12) $t^1(M)$ is Hodge-Witt so $H^0(\underline{\underline{s}}(t^1(M)) = 0$ implies $t^1(M) = 0$. This gives iii). Finally, iv) follows from Corollary 1.4.1.

The notion opposite to $\underline{\underline{s}}$-acyclic objects is the notion of Mazur-Ogus objects.

<u>Definition</u> 4.3. *A diagonal complex* M *is said to be Mazur-Ogus if* $\mathrm{Hom}_{\Delta}(N,M) = \mathrm{Hom}_{\Delta}(M,N) = 0$ *for all* $\underline{\underline{s}}$*-acyclic objects* N . *The category of Mazur-Ogus objects will be denoted* Δ_{MO} .

Also for Mazur-Ogus objects is there a crucial result.

<u>Lemma</u> 4.4. *Suppose* $M \in \Delta_{MO}$. *Then* $H^i(R_1 \otimes^L_R M)^j = 0$ *if* $i+j \neq 0$ *and* M *is without* $\underline{\underline{s}}$*-torsion.*

Suppose $H = 1$. Clearly $\underline{\underline{U}}^i_0$ is $\underline{\underline{s}}$-acyclic for all i . By (0:4.3) we get that $H^i(R_1 \otimes^L_R M)^j = 0$ if $i+j = 1$ or -1 and combining this with (1.4 ii) we get $H^i(R_1 \otimes^L_R M)^j = 0$ if $i+j \neq 0$. The spectral sequence (cf. (0:6.2.1) $H^i(R_1 \otimes^L_R M)^j \Longrightarrow H^{i+j}(k \otimes^L_W \underline{\underline{s}}(M))$ shows that $k \otimes^L_W \underline{\underline{s}}(M)$ is concentrated in degree 0 and the triangle $\longrightarrow \underline{\underline{s}}(M) \xrightarrow{p} \underline{\underline{s}}(M) \longrightarrow k \otimes^L_W \underline{\underline{s}}(M) \longrightarrow$ shows that $\underline{\underline{s}}(M)$ is concentrated in degree 0 and that $H^0(\underline{\underline{s}}(M))$ is torsion free. When $H \neq 1$ we replace $\underline{\underline{U}}^i_0$ by $U^i_0[H]$.

We have now come to the major result on the structure of Δ .

<u>Theorem</u> 4.5. *For any* $M \in \Delta$ *there exists two (canonical) functorial filtrations characterized by* i) *resp.* ii)

(4.5.1) $$0 \subseteq F^1 \subseteq F^2 \subseteq F^3 = M$$

(4.5.2) $$0 \subseteq G^1 \subseteq G^2 \subseteq G^3 = M$$

such that

i) F^1 *is the largest* s-0-*torsion subobject of* M *and* F^3/F^2 *is the largest* $\underline{\underline{s}}$-1-*torsion quotient object of* F^3/F^1 .

ii) G^3/G^2 *is the largest* $\underline{\underline{s}}$-1-*torsion quotient object of* M *and* G^1 *is the largest* $\underline{\underline{s}}$-0-*torsion subobject of* G^2 .

iii) $G^1 = F^1 \cap G^2$; $F^2 = F^1 + G^1$.

iv) $F^2/F^1 = G^2/G^1$ *is Mazur-Ogus and* $F^1/G^1 = F^2/G^2$ *is* $\underline{\underline{s}}$*-acyclic.*

v) $F^2/G^1 = F^2/F^1 \oplus F^1/G^1$.

The existence of F^1, F^2, G^1 and G^2 follows from (0:1.1), (1.5) (1.11 iv) and (4.2). Now, v) follows from iii) so we are left with iii) and iv). Let us first show that F^2/F^1 and G^2/G^1 are Mazur-Ogus. If F^2/F^1 is not then there exists some $\underline{\underline{s}}$-acyclic N and either a non-zero morphism $F^2/F^1 \longrightarrow N$ or a non-zero morphism $N \longrightarrow F^2/F^1$. The image of these would, by (4.2), be $\underline{\underline{s}}$-1-torsion and $\underline{\underline{s}}$-0-torsion respectively which contradicts the definition of F^2 resp. F^1 . Similarly for G^2/G^1 .

Let us now show that $F^1/G^2 \cap F^1$ is $\underline{\underline{s}}$-acyclic and that $G^1 = G^2 \cap F^1$. By (4.2) $\mathrm{Im}(F^1 \longrightarrow G^3/G^2)$ is $\underline{\underline{s}}$-0-torsion and $\underline{\underline{s}}$-1-torsion i.e. $\underline{\underline{s}}$-acyclic. This image is isomorphic to $F^1/G^2 \cap F^1$ which therefore is $\underline{\underline{s}}$-acyclic and hence $\underline{\underline{s}}(G^2 \cap F^1) \longrightarrow \underline{\underline{s}}(F^1)$ is an isomorphism so $G^2 \cap F^1$ is $\underline{\underline{s}}$-0-torsion. By the definition of G^1 we have $G^2 \cap F^1 \subseteq G^1$ but by the definition of F^1 , $G^1 \subseteq F^1$ and by definition $G^1 \subseteq G^2$ so $G^1 \subseteq G^2 \cap F^1$ and consequently $G^1 = G^2 \cap F^1$. As the equalities $F^2/F^1 = G^2/G^1$ and $F^1/G^1 = F^2/G^2$ follow from iii) and as we have already proved that F^2/F^1 is Mazur-Ogus and that $F^1/G^1 = F^1/G^2 \cap F^1 = \mathrm{Im}(F^1 \longrightarrow G^3/G^2)$ is $\underline{\underline{s}}$-acyclic we are left with the equality $F^2 = F^1 + G^2$ or what amounts to the same that $G^2/G^1 \longrightarrow F^2/F^1$ is an epimorphism. Now, G^3/G^2 is $\underline{\underline{s}}$-1-torsion and by lemma 4.4 G^2/G^1 is without $\underline{\underline{s}}$-torsion. Hence by (4.2.1) and (1.4) $H^0(\underline{\underline{s}}(G^3/G^1)) = H^0(\underline{\underline{s}}(G^2/G^1))$ and it is torsion free and $H^1(\underline{\underline{s}}(G^3/G^1)) = H^1(\underline{\underline{s}}(G^3/G^2))$. Using (4.2.1) one more time we see that $\text{tors-}H^0(\underline{\underline{s}}(M)) = H^0(\underline{\underline{s}}(G^1))$, $H^0(\underline{\underline{s}}(M))/\text{tors} = H^0(\underline{\underline{s}}(G^2/G^1))$ and $H^1(\underline{\underline{s}}(M)) = H^1(\underline{\underline{s}}(G^3/G^2))$. In the same way $\text{tors-}H^0(\underline{\underline{s}}(M)) = H^0(\underline{\underline{s}}(P^1))$, $H^0(\underline{\underline{s}}(M))/\text{tors} = H^0(\underline{\underline{s}}(F^2/F^1))$ and $H^1(\underline{\underline{s}}(M)) = H^1(\underline{\underline{s}}(F^3/F^2))$. In particular we see that $G^2/G^1 \longrightarrow F^2/F^1$ induces isomorphisms on $H^0(\underline{\underline{s}}(-))$ and on $H^1(\underline{\underline{s}}(-))$ (both are zero in that case)

so $\underline{\underline{s}}(G^2/G^1) \longrightarrow \underline{\underline{s}}(F^2/F^1)$ is an isomorphism and the cokernel of $G^2/G^1 \longrightarrow F^2/F^1$ is $\underline{\underline{s}}$-acyclic and hence 0 as F^2/F^1 is Mazur-Ogus.

Let us also write down the formulas obtained in the last part of the argument.

<u>Corollary</u> 4.5.3.

$$H^0(\underline{\underline{s}}(F^1)) = \text{tors-}H^0(\underline{\underline{s}}(M)) = H^0(\underline{\underline{s}}(G^1))$$

$$H^0(\underline{\underline{s}}(F^2/F^1)) = H^0(\underline{\underline{s}}(M))/\text{tors} = H^0(\underline{\underline{s}}(G^2/G^1))$$

$$H^1(\underline{\underline{s}}(F^3/F^2)) = H^1(\underline{\underline{s}}(M)) = H^1(\underline{\underline{s}}(G^3/G^2)) .$$

II

F-gauge structures

1. As we have seen, if $M \in \Delta$ then $\underline{\underline{s}}(M)$ is almost but sometimes not completely concentrated in degree 0. By changing Δ a little we can get the associated simple complex to be concentrated in degree 0. Define a functorial filtration on $\Delta \ldots \subseteq M^i \subseteq M^{i+1} \subseteq \ldots M$ by $M^i = 0$ if $i < 0$, $M^i = M$ if $i > 0$ and $M^0 = G^2$ (cf. Thm. I:4.5). By (0:1.1) and (I:4.2) M^0 is an idempotent radical.

<u>Definition</u> 1.1. *Let* $(G^{\leq 0}, G^{\geq 0})$ *be the* t-*structure associated to the radical filtration* M^i *and* $G := G^{\leq 0} \cap G^{\geq 0}$, $\tau^g_{\leq i}$, $\tau^g_{\geq i}$ *the* G-*truncations and* $H^i_g(-) := \tau^g_{\leq i} \circ \tau^g_{\geq i}(-)[i]$.

<u>Proposition</u> 1.2. *Let* $M \in D^b_c(R)$. *Then* M *has* G-*amplitude* $[m,n]$ *iff* $\underline{\underline{s}}(M)$ *has amplitude* $[m,n]$ *and* $H^i(R_1 \otimes^L_R M)^j = 0$ *unless* $m-2 \leq i+j \leq n$.

<u>Proof</u> : We may assume $H = 1$. Let first $M \in G$. By construction we have an exact sequence

$$0 \longrightarrow \widetilde{H}^{-1}(M)[-1] \longrightarrow M \longrightarrow \widetilde{H}^0(M) \longrightarrow 0$$

such that $\widetilde{H}^0(M)$ has no $\underline{\underline{s}}$-1-torsion quotients and $\widetilde{H}^{-1}(M)$ is $\underline{\underline{s}}$-1-torsion. By (I:4.5.1) $H^1(\underline{\underline{s}}(\widetilde{H}^0(M))) = 0$ so we see that $\underline{\underline{s}}(M)$ has amplitude $[0,0]$. Furthermore, by (I:4.2) the image of any morphism $\widetilde{H}^0(M) \longrightarrow \underline{\underline{U}}^i_0$ would be $\underline{\underline{s}}$-1-torsion and hence zero by assumption. This together with (0:4.3) shows that $H^i(R_1 \otimes^L_R \widetilde{H}^0(M))^j = 0$ if $i+j = 1$ and then (I:1.4 ii) shows that $H^i(R_1 \otimes^L_R M)^j = 0$ if $i+j \neq -2,-1,0$. Dévissage now gives the only if-part. Suppose now that $\underline{\underline{s}}(M)$ has amplitude $[m,n]$ and $H^i(R_1 \otimes^L_R M)^j = 0$ unless

$m-2 \leq i+j \leq n$ but that $H_g^r(M) \neq 0$ and $H_g^i(M) = 0$ for $i < r$ for some $r < m$.
By the only if-part

$$\underline{\underline{s}}(H_g^r(M)) = H^0(\underline{\underline{s}}H_g^r(M)) = H^r(\underline{\underline{s}}(M)) = 0 \text{ and}$$

by the only if-part and the spectral sequence

$$H^i(R_1 \otimes_R^L H_g^j(M))^k \Longrightarrow H^{i+j}(R_1 \otimes_R^L M)^k$$

(cf. (0:1.5)) we see that $H^i(R_1 \otimes_{R_1}^L H_g^r(M))^j = 0$ if $i+j \leq -1$. By (I:4.2 iv) $\widetilde{H}^i(H_g^r(M))$ are $\underline{\underline{s}}$-acyclic for all i and non-zero only for $i = 0, -1$ by definition of G). By (I:1.4.1 ii) $H^i(R_1 \otimes \widetilde{H}^{-1}(H_g^r(M)))^j = 0$ if $i+j = 0$ and this implies (I:1.4 ii) that the spectral sequence

$$H^i(R_1 \otimes_R^L \widetilde{H}^{-1}(H_g^r(M)))^j \Longrightarrow H^{i+j}(k \otimes_W^L \underline{\underline{s}}(\widetilde{H}^{-1}(H_g^r(M)))) = 0$$

degenerates and hence that $R_1 \otimes_R^L \widetilde{H}^{-1}(H_g^r(M)) = 0$ and (0:4) that $\widetilde{H}^{-1}(H_g^r(M)) = 0$. Hence $H_g^r(M) \in \Delta$, is $\underline{\underline{s}}$-acyclic and $H^i(R_1 \otimes_R^L H_g^r(M))^j = 0$ if $i+j = -1$ that is, (0:4.3), $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^i, H_g^r(M)) = 0$ for all i. The following lemma then shows that $H_g^r(M) = 0$.

<u>Lemma</u> 1.2.1. *The category of* $\underline{\underline{s}}$*-acyclic diagonal dominoes is an abelian category all of whose objects have finite length and its simple objects are the* $\underline{\underline{U}}_0^i$, *if* $H = 1$.

Indeed, it is clear that $\underline{\underline{U}}_0^i$ is $\underline{\underline{s}}$-acyclic. Let $M \hookrightarrow \underline{\underline{U}}_0^i$ be an $\underline{\underline{s}}$-acyclic non-trivial subobject. By shifting we may assume that $i = 0$. By (I:1.5.1) $M \in \Delta \cap R\text{-mod}$ and $M^0 \longrightarrow (\underline{\underline{U}}_0)^0$ is injective. Hence M is a 1-dimensional domino so isomorphic to $\underline{\underline{U}}_j$ for some j and $\underline{\underline{s}}(M) = 0$ gives $j = 0$ and then, by (0:4.2), $M \longrightarrow \underline{\underline{U}}_0$ is an isomorphism. Now (I:4.2 iv) shows that the $\underline{\underline{s}}$-acyclic diagonal complexes form an abelian subcategory of Δ. Then (I:1.11), (I:4.2 iv) and the fact that $\underline{\underline{s}}$-acyclic objects are <u>a fortiori</u> $\underline{\underline{s}}$-1-torsion imply that

the $\underline{\underline{s}}$-acyclic objects form an artinian category so being both artinian and noetherian every object has finite length. We have seen that $\underline{\underline{U}}_0^i$ are simple objects. Let M be some simple object. To show that it is isomorphic to some $\underline{\underline{U}}_0^i$ it suffices to show that $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^i, M)$ or $\mathrm{Hom}_\Delta(M, \underline{\underline{U}}_0^i)$ is non-zero for some i. If this were not the case then by (0:4.3) $H^i(R_1 \otimes_R^L M)^j = 0$ for $i+j = -1$ or 1 and by (I:1.4 ii) the spectral sequence

$$H^i(R_1 \otimes_R^L M)^j \Longrightarrow H^{i+j}(k \otimes_W^L \underline{\underline{s}}(M)) = 0$$

would degenerate so $R_1 \otimes_R^L M$ would be zero and (0:4) would give that $M = 0$ contrary to the assumption that M is simple.

We have thus proved that the M of the proposition has G-amplitude $[m,\infty[$. In exactly the same way we prove that M has G-amplitude $]-\infty,n]$.

<u>Corollary</u> 1.2.2. i) $\underline{\underline{s}}(-)$ *is G-exact*

ii) $(-) \hat{*}_R^L (-)$ *has G-amplitude* $[-2,0]$

iii) $\mathrm{RHom}_R^!(-,-)$ *has G-amplitude* $[0,2]$.

Indeed, i) is obvious and ii) and iii) follows from (0:3.3, 3.3.1).

2. Our purpose is now to show that $(G^{\leq 0}, G^{\geq 0})$ admits a more concrete description.

<u>Definition</u> 2.1. i) *An F-gauge structure is a graded W-module* $M = \bigoplus_{i\in\mathbb{Z}} M^i$ *together with linear mappings* $\tilde{F}$ *and* $\tilde{V}$ *of degree* 1 *and* -1 *respectively s.t.* $\tilde{F}\tilde{V} = \tilde{V}\tilde{F} = p$ *and a σ-linear isomorphism* $\tau : M^{\infty} := \varinjlim(\dots \xrightarrow{\tilde{F}} M^i \xrightarrow{\tilde{F}} M^{i+1} \xrightarrow{\tilde{F}} \dots) \longrightarrow M^{-\infty} := \varinjlim(\dots \xrightarrow{\tilde{V}} M^i \xrightarrow{\tilde{V}} M^{i-1} \xrightarrow{\tilde{V}} \dots)$.

ii) Let I be an interval of $\mathbb{Z}$. An F-gauge structure is of level I if for every $n \in \mathbb{Z}$ below I $\tilde{V} : M^{n+1} \longrightarrow M^n$ is an isomorphism and for every n above I $\tilde{F} : M^{n-1} \longrightarrow M^n$ is an isomorphism. If $I = [0,n]$ $n \in \mathbb{N}$ we will also say that the F-gauge structure is of level n. If $I = [0,\infty[$ we will also say that the F-gauge structure is effective.

iii) The category, with the evident morphisms, of F-gauge structures (of level I) will be denoted F-g-str(-I).

Remark : i) It may be more reasonable to require instead an isomorphism $\tau : "\varinjlim"(M^i,\tilde{F}) \longrightarrow "\varinjlim"(M^i,\tilde{V})$. We will only consider F-gauge structures of finite level where there is no difference.

ii) Let $I = [m,n]$. On $\underline{M} = \bigoplus_{i\in I} M^i$ we may define endomorphisms $\underline{F}$ and $\underline{V}$ by $\underline{F} : M^i \longrightarrow M^\infty \xrightarrow{\tau} M^{-\infty} \xrightarrow{\sim} M^m \xrightarrow{\tilde{F}^{i-m}} M^i$ and $\underline{V} : M^i \longrightarrow M \xrightarrow{\tau^{-1}} M \xrightarrow{\sim} M^n \xrightarrow{\tilde{V}^{n-i}} M^i$. In this way we give $\underline{M}$ a structure of module over the W-ring generated by F and V and relations $\underline{F}\underline{V} = \underline{V}\underline{F} = p^{n-m}$ $Fa = a^\sigma F$ $aV = Va^\sigma$ for $a \in W$. Just as it sometimes is convenient to regard $\hat{R}$-modules rather than R-modules it is sometimes convenient to consider F-gauge structures of level I where $\underline{M}$ also has been given a structure of module over the p-adic completion of this ring. If in all that is to come we replace R by $\hat{R}$ and add this extra condition to our F-gauge structures then all our results will remain true with only trivial modifications of the proofs.

Examples : i) A virtual F-crystal is a triple $(U,\underline{F},N)$ where U is a finite dimensional K-vector space $K := \mathrm{Frac}(W)$, $\underline{F}$ a σ-linear automorphism of U and N a W-submodule of U of maximal rank (i.e. $KN = U$). We will say that the virtual F-crystal is of finite type if N is a finitely generated W-module. Suppose now that M is a torsion free F-gauge structure such that $M^{-\infty} \otimes_W K$ is a finite dimensional K-module. The relations $\tilde{F}\tilde{V} = p$ show that $M^\infty \otimes_W K = \varinjlim(M^i \otimes K,\tilde{F}) = M^0 \otimes_W K$ and similarly for $M^{-\infty} \otimes_W K$. Hence we get a virtual F-crystal where $U := M^{-\infty} \otimes_W K$, $\underline{F}$ is $M^{-\infty} \otimes_W K = M^0 \otimes_W K = M^\infty \otimes_W K \xrightarrow{\tau} M^{-\infty} \otimes_W K$ and $N = M^{-\infty} \hookrightarrow M^{-\infty} \otimes_W K$. The images of M^i in U form a decreasing filtration of U with the properties

a) $pM^i \subseteq M^{i+1}$

b) $N = \bigcup_i M^i$

c) F maps $\bigcup_i p^{-i}M^i$ into and onto $\bigcup_i M^i$.

Conversely, any virtual F-crystal (U,F,N) with a decreasing filtration fulfilling a)-c) gives an F-gauge structure by putting $M^i = M^i$, $\widetilde{V}$ the inclusion, $\widetilde{F}$ multiplication by p and τ the isomorphism obtained by condition c). Furthermore, the virtual F-crystal is of finite type iff $\bigcup_i M^i$ is of finite type in which case $\bigcup_i p^{-i}M^i$ is also of finite type and the level is finite. Conversely, if the level is finite and one (and hence all) of the M^i are of finite type then the associated virtual F-crystal is of finite type.

The forgetful functor $(M,\widetilde{F},\widetilde{V},\tau) \longmapsto (U,\underline{F},N)$ has a right adjoint. Namely we put for $(U,\underline{F},N)$ a virtual F-crystal $M^i = (\underline{F}^{-1}(p^iN)) \cap N$. Then $\{M^i\}$ is decreasing, $\bigcup_i M^i = (\underline{F}^{-1}(\bigcup_i p^iN)) \cap N = \underline{F}^{-1}V \cap N = N$ and $\underline{F}(\bigcup_i p^{-i}M^i) = \underline{F}(\underline{F}^{-1}N \cap \bigcup_i p^{-i}N) = \underline{F}(\underline{F}^{-1}N \cap V) = N$. Furthermore $\{M^i\}$ is clearly the largest filtration fulfilling a)-c) so we get our right adjoint. We will denote this functor Hodge(-) and call the obtained F-gauge structure the Hodge F-gauge structure associated to a virtual F-crystal.

Note that the Hodge F-gauge structure is of level I iff $p^{-n-1}\underline{F}N \subseteq N$ for all n below I and $p^{-i+1}\underline{F}N \supseteq N$ for all i above I . (This is left to the reader). In particular a virtual F-crystal of finite type whose Hodge F-gauge structure is effective is just an F-crystal in the ordinary sense.

ii) An F-gauge structure of level 1 is just an R^0-module. Indeed, $M^{-\infty} = M^0$ and $M^1 = M^\infty$ so we can put $F : M^0 \xrightarrow{\widetilde{F}} M^1 = M^\infty \xrightarrow{\tau} M^{-\infty} = M^0$ and $V : M^0 = M^{-\infty} \xrightarrow{\tau^{-1}} M^\infty = M^1 \xrightarrow{\widetilde{V}} M^0$. In particular we can regard Dieudonné modules as a subcategory of F-g-str.

iii) Consider an F-gauge structure M such that M is of finite length as a W-module. Then M^i is zero for all but a finite number of i so $M^{\infty} = M^{-\infty} = 0$ so we get a homomorphism $W \longrightarrow \mathrm{End}(M)$. As we shall see these M correspond to $\underline{\underline{s}}$-acyclic diagonal complexes.

iv) Fix some finite level I. The F-gauge structures of level $I = [m,n]$ may be identified with $M' = \bigoplus_{i\in I} M^i$ together with maps $F' : M^i \longrightarrow M^{i+1}$ $m \leq i < n$ $F' : M^n \longrightarrow M^m$, $V' : M^{i+1} \longrightarrow M^i$ $m \leq i < n$ $V' : M^m \longrightarrow M^n$ with F', V' W-linear except for $F' : M^n \longrightarrow M^m$ which is σ-linear and $V' : M^m \longrightarrow M^n$ σ^{-1}-linear and $F'V' = V'F' = p$ except for $V'F' : M^n \longrightarrow M^n$ and $F'V' : M^m \longrightarrow M^m$ which are the identity. Whenever convenient we will use this description.

v) We continue to fix $I = [m,n]$. The functor $T^r : \text{F-g-str-}I \longrightarrow \mathrm{Ab}$ given $T^r(M) = M^r$ is represented by a necessarily projective F-gauge structure G_r. An explicit description is given as follows $\bigoplus_{i\in I} G_r^i = \sum_{s\in\mathbb{N}} WF'^s\alpha_r + \sum_{s\in\mathbb{N}} WV'^s\alpha_r$ where $\alpha_r \in G_r^r$ and $\mathrm{Hom}(G_r, M) \longrightarrow M^r$ is given by $\varphi \longmapsto \varphi(\alpha_r)$. As $\{T^r(-)\}$ is a conservative set of functors the G_r for $r \in I$ form a set of projective generators for F-g-str-I.

3. We will now to an R-module associate a complex of F-gauge structures (cf. [Ny]).

Definition 3.1. i) *Let* M *be an* R-*module. Then* $S(M)$ *is the following complex of* F-*gauge structures*

a) $S(M)^i := \underline{\underline{s}}(M(\varphi^i))$ *with* $\varphi^i(n) = -\delta_{in}$.

b)
$$\begin{array}{ccccccccccccc}
S(M)^i & : \dots & \xrightarrow{d} & \sigma_*M^{i-2} & \xrightarrow{d} & \sigma_*M^{i-1} & \xrightarrow{dV} & M^i & \xrightarrow{d} & M^{i+1} & \xrightarrow{d} & \dots \\
\Big\downarrow{\scriptstyle\tilde F} & & & \Big\| & & \Big\| & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle p} & & \\
S(M)^{i+1} & : \dots & \xrightarrow{d} & \sigma_*M^{i-2} & \xrightarrow{d} & \sigma_*M^{i-1} & \xrightarrow{d} & \sigma_*M^i & \xrightarrow{dV} & M^{i+1} & \xrightarrow{d} & \dots
\end{array}$$

$$\begin{array}{cccccccccccc}
S(M)^i & :\ldots & \xrightarrow{d} & \sigma_*M^{i-3} & \xrightarrow{d} & \sigma_*M^{i-2} & \xrightarrow{d} & \sigma_*M^{i-1} & \xrightarrow{dV} & M^i & \xrightarrow{d} & \ldots \\
\downarrow{\scriptstyle \tilde{V}} & & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle V} & & \| & & \\
S(M)^{i+1} & :\ldots & \xrightarrow{d} & \sigma_*M^{i-3} & \xrightarrow{d} & \sigma_*M^{i-2} & \xrightarrow{dV} & M^{i-1} & \xrightarrow{d} & M^i & \xrightarrow{d} & \ldots
\end{array}$$

c) $S(M)^{\infty} = \sigma_*\underline{\underline{s}}(M)$, $S(M)^{-\infty} = \underline{\underline{s}}(M)$ *and* τ *is the identity.*

ii) *Let* M *be a complex of* R*-modules. Put* $\underline{\underline{S}}(M)$ *the simple complex (using sums) associated to the double complex* $S(M)$.

We have thus a functor from complexes of R-modules to complexes of F-gauge structures. This functor clearly preserves quasi-isomorphisms and mapping cones so it extends to a triangulated functor from $D(R)$ to $D(F\text{-g-str})$. The formulas used to define $\tilde{F}$ and $\tilde{V}$ also show that it takes $D(R\text{-}I)$ to $D(F\text{-g-str-}I)$ for any interval I.

<ins>Definition</ins> 3.2. *Let* $M \in D(F\text{-g-str-}I)$ *for some finite interval* I. *The completion* $\hat{M}$ *of* M *is* $R\varprojlim\{W/p^n \otimes^L_W M\}$ *with its obvious structure of complex of* F*-gauge structures of level* I. *An* $M \in D(F\text{-g-str-}I)$ *is said to be complete if the canonical morphism* $M \longrightarrow \hat{M}$ *is an isomorphism.*

One shows without difficulty that $\hat{M} = \hat{\hat{M}}$.

<ins>Proposition</ins> 3.3. *The functor* $\underline{\underline{S}} : D(R\text{-}I) \longrightarrow D(F\text{-g-str-}I)$ *commutes with completion and hence takes complete complexes to complete complexes.*

<ins>Proof</ins> : This follows from $\underline{\underline{s}}((\hat{-})) = \underline{\underline{s}}(-)$ and $(\hat{-})(\varphi) = (\widehat{(-)(\varphi)})$ (cf. (0:2.1)).

<ins>Proposition</ins> 3.4. *Let* $M \in D(R\text{-}I)$. *Then there exists a commutative up to sign diagram of distinguished triangles*

$$
(3.4.1)\qquad
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \underline{\underline{S}}(M)^{i} \arrow[r, "\widetilde{F}"] \arrow[d, "\widetilde{V}"] & \underline{\underline{S}}(M)^{i+1} \arrow[r] \arrow[d, "\widetilde{V}"] & t_{\geq i+1}\mathrm{Hod}(M) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \underline{\underline{S}}(M)^{i-1} \arrow[r, "\widetilde{F}"] \arrow[d] & \underline{\underline{S}}(M)^{i} \arrow[r] \arrow[d] & t_{\geq i}\mathrm{Hod}(M) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \sigma_{*}\tau_{\leq i-1}\mathrm{Hod}(M) \arrow[r] \arrow[d] & \sigma_{*}\tau_{\leq i}\mathrm{Hod}(M) \arrow[r] \arrow[d] & (R_1 \otimes^{L}_{R} M)^{i} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}
$$

where the lowest and the right most triangles are those of (0:3.4).

To define (3.4.1), we may assume that M is R_1 - and Z_1 - acyclic and that p is injective on M. We will define our morphisms already on the level of double complexes so we may assume that M is just an R-module. We then define

$$S(M)^{i}/\widetilde{F} \longrightarrow t_{\geq i}\mathrm{Hod}(M) \quad \text{by (cf. [Ny:1.3])}$$

$$
\begin{tikzcd}
0 \arrow[r] & \sigma_{*}M^{i-1}/F \arrow[r, "dV"] \arrow[d] & M^{i}/p \arrow[r, "d"] \arrow[d] & \cdots \\
 & 0 \arrow[r] & M^{i}/VM^{i}+dVM^{i} \arrow[r] & \cdots
\end{tikzcd}
$$

and $S(M)^{i}/\widetilde{V} \longrightarrow \tau_{\leq i}\mathrm{Hod}(M)$ by

$$
\begin{tikzcd}
\cdots \arrow[r, "d"] & \sigma_{*}M^{i-1}/P \arrow[r, "dV"] \arrow[d] & M^{i}/V \arrow[r] \arrow[d, "F"] & 0 \\
\cdots \arrow[r, "d"] & \sigma_{*}M^{i-1}/V+dV \arrow[r, "d"] & Z_1M^{i} \arrow[r] & 0 .
\end{tikzcd}
$$

To show that the triangles are distinguished we reduce by a limit argument to the case where M is bounded in all directions and then by dévissage and shift to $M = R$ in which case we compute directly (or use the argument of [N:1]).

Corollary 3.4.1. *The functor* $\underline{\underline{S}} : D(R\text{-}I) \longrightarrow D(F\text{-}g\text{-}str\text{-}I)$, I *a finite interval, is conservative on the category of complete complexes.*

Indeed, if $\underline{\underline{S}}(M) = 0$ then we have seen that $R_1 \otimes^L_R M = 0$ and as M is complete this implies that $M = 0$ (0:4).

4. We will now go in the other direction. Starting from an F-gauge structure we will construct a complex of R-modules. Recall that $C(R\text{-}I)$, I a finite interval, denotes the abelian category of complexes of R-modules of level I. The functor $Z^0\underline{\underline{S}}(-) : C(R\text{-}I) \longrightarrow F\text{-}g\text{-}str\text{-}I$ taking an $X \in C(R\text{-}I)$ to $\operatorname{Ker} D : \underline{\underline{S}}(X)^0 \longrightarrow \underline{\underline{S}}(X)^0$, where D is the total differential, commutes with inverse limits and hence has a left adjoint t. Here is an explicit description of t. We let δ denote the differential of an R-complex.

Definition 4.1. *Let* $M \in F\text{-}g\text{-}str$. *Then* $t(M)$ *is the* R-*complex of level* I *generated by the* $\beta_i \otimes m$ $i \in I$, $m \in M$ *with* $\beta_i \otimes m$ *in module degree* i *and complex degree* $-i$ *and with relations*

i) $\forall \lambda \in W$ $m_1, m_2 \in M^j$

$$\beta_i \otimes (\lambda m_1 + m_2) = \lambda(\beta_i \otimes m_1) + \beta_i \otimes m_2 \qquad \text{if} \quad i > j-1$$

$$\beta_i \otimes (\lambda m_1 + m_2) = \lambda^{\sigma^{-1}}(\beta_i \otimes m_1) + \beta_i \otimes m_2 \qquad \text{if} \quad i \leq j-1 .$$

ii) *If* $m_j \in M^j$ *then*

$$\delta(\beta_{i+1} \otimes m_j) = d(\beta_i \otimes m_j) \qquad \text{if} \quad i \neq j-1$$

$$\delta(\beta_j \otimes m_j) = dV(\beta_{j-1} \otimes m_j)$$

iii) *If* $m_j \in M^j$ *then* $\forall i \in I$

$$\beta_i \otimes \tilde{F}m_j = \begin{cases} \beta_i \otimes m_j & i < j \\ F\beta_i \otimes m_j & i = j \\ p\beta_i \otimes m_j & i > j \end{cases}$$

$$\beta_i \otimes \tilde{V}m_j = \begin{cases} \beta_i \otimes m_j & j < i-1 \\ V\beta_i \otimes m_j & j = i-1 \\ p\beta_i \otimes m_j & j > i-1 \end{cases} .$$

iv) *If* $m_i \in M^i$, $m_j \in M^j$ *and the image of* m_i *in* $M^{-\infty}$ *equals* τ *applied to the image of* m_j *in* M^{∞} *then* $\forall r \in I$

$$\beta_r \otimes m_i = \beta_r \otimes m_j .$$

I leave to the reader to verify that

$$M \longrightarrow Z^0\underline{\underline{S}}(t(M))$$

$$m \longmapsto \sum_{i \in I} \beta_i \otimes m$$

is the adjunction unit for an adjunction between t and $Z^0\underline{\underline{S}}$.

Lemma 4.2. *For all* $r \in I = [m,n]$

$$t(G_r) = \bigoplus_{m \leq i < n} R(\beta_i \otimes \alpha_r) \oplus W_\sigma[F,F^{-1}](\beta_n \otimes \alpha_r)$$

where $R(\beta_i \otimes \alpha_r)$ *resp.* $W_\sigma[F,F^{-1}](\beta_n \otimes \alpha_r)$ *denotes the free* R*-resp.* $W_\sigma[F,F^{-1}]$*-module on* $\beta_i \otimes \alpha_r$ *resp.* $\beta_n \otimes \alpha_r$.

Proof : The relations in iii) and iv) show that $t(G_r) = \sum_{m \leq i < n} R\beta_i \otimes \alpha_r + W_\sigma[F,F^{-1}]\beta_n \otimes \alpha_r$. Using the relations in ii) one defines an R-complex of level I-structure on $N = \sum_{m \leq i < n} R(\beta_i \otimes \alpha_r) \oplus W_\sigma[F,F^{-1}](\beta_n \otimes \alpha_r)$ such that

we have a map $G_r \to Z^0\underline{\underline{S}}(N)$ sending α_r to $\sum \beta_i \otimes \alpha_r$ hence we get a morphism $t(G_r) \to N$ taking the class of $\beta_i \otimes \alpha_r$ $m \leq i \leq n$ to $\beta_i \otimes \alpha_r$ and hence there are non-trivial relations.

Now, $Z^0\underline{\underline{S}}(-)$ is the composite of $\underline{\underline{S}}(-)$ and $Z^0 : C(\text{F-g-str}) \to \text{F-g-str}$ which is right adjoint to the inclusion. As $\underline{\underline{S}}(-)$ commutes with inverse limits it also has a left adjoint $T : C(\text{F-g-str})) \to C(\text{R-I}))$. On the other hand we can extend t as usual to complexes, we take the associated simple complex of the natural double complex. For this extension we get a natural transformation $\mathrm{id} \to \underline{\underline{S}}(t(-))$ hence getting by adjunction $T \to t$. There are many ways of seeing that this is an equivalence. One is to reduce to bounded complexes as both commute with direct limits, then using the fact that $T \circ \mathrm{incl} : \text{F-g-str} \to C(\text{F-g-str}) \to C(\text{R-I})$ equals t by transitivity of adjoints and then, starting from this argue by induction on the length of the complex X. We have $0 \to t_{\geq i}X \to X \to t_{<i}X \to 0$ which gives exact sequences

$$\begin{array}{ccccccccc}
 & & T(t_{\geq i}X) & \to & T(X) & \to & T(t_{<i}X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & t(t_{\geq i}X) & \to & t(X) & \to & t(t_{<i}X) & \to & 0 .
\end{array}$$

By induction $T(t_{\geq i}X) \to t(t_{\geq i}X)$ is an isomorphism so $T(t_{\geq i}X) \to T(X)$ is a monomorphism and then we use five lemma.

In particular we have adjunction morphisms

$$(4.3) \qquad t(\underline{\underline{S}}(-)) \to \mathrm{id} \qquad \mathrm{id} \to \underline{\underline{S}}(t(-)) .$$

We then right derive t to get Lt defined on $D^-(\text{F-g-str-I})$ and then get complete to get $L\hat{t} := \widehat{(Lt)}$. As $\underline{\underline{S}}$ commutes with completion we get for $M \in D^-(\text{F-g-str-I})$ and $N \in D^-(\text{R-I})$

$$(4.4) \qquad \varphi : L\hat{t}(\underline{\underline{S}}(N)) \to \hat{N} \qquad \psi : \hat{M} \to \underline{\underline{S}}(L\hat{t}(M)) .$$

Theorem 4.5. *For* $M \in D^{-}(F\text{-}g\text{-}str\text{-}I)$ *and* $N \in D^{-}(R\text{-}I)$ *the maps in* (4.4) *are isomorphisms.*

Proof : As $\underline{\underline{S}}$ is conservative on complete complexes to show that φ is an isomorphism it suffices to show that $\underline{\underline{S}}(\varphi)$ is an isomorphism. This, however, is the case of ψ for $M = \underline{\underline{S}}(N)$ so we are reduced to showing that ψ is an isomorphism and it is enough to show that $M \longrightarrow \underline{\underline{S}}(Lt(M))$ is an isomorphism by (2.3). As t and $\underline{\underline{S}}$ commute with direct sums we may by dévissage reduce to $M = G_r$ for some $r \in I$. What needs to be shown is that for every j $G_r^j \longrightarrow \underline{\underline{S}}(t(G_r))^j$, is a quasi isomorphism. Because both G_r and $\underline{\underline{S}}(t(G_r))$ are of level I we may assume that $j \in I$. Let us fix some notation. Put $t = m = n+1$ where $I = [m,n]$. We use the description $G_r = \sum_{i \in \mathbb{N}} F'^{i}\alpha_r \oplus \sum_{i \in \mathbb{N}} V'^{i}\alpha_r$ and put $\underline{F} := F'^{t}$, $\underline{V} := V'^{t}$ $\varepsilon := F'^{j-r+m}\alpha_r$ $\varphi := V'^{r-j+n}\alpha_r$ where $m = 0$ if $j-r \geq 0$ and $=t$ if not and $n = 0$ if $r-j \geq 0$ and $=t$ if not. Then $G_r^j = \sum_{i \in \mathbb{N}} \underline{V}^{i}\varphi \oplus \sum_{i \in \mathbb{N}} \underline{F}^{i}\varepsilon$. Furthermore, (4.1, iii)-iv)) shows that

$$(4.5.1) \qquad \beta_m \otimes \underline{F}^k\varepsilon = \begin{cases} F^k(\beta_m \otimes \alpha_r) & \text{if } j \geq r \\ F^{k+1}(\beta_m \otimes \alpha_r) & \text{if } j < r \end{cases}$$

$$(4.5.2) \qquad \beta_m \otimes \underline{V}^k\varphi = \begin{cases} p^{(n-m)k+r-j}V^k(\beta_m \otimes \alpha_r) & \text{if } j \leq r \\ p^{(n-m)(k+1)-(r-j)}V^{k+1}(\beta_m \otimes \alpha_r) & \text{if } j > r \end{cases}$$

By extension of k we may assume that it is infinite (which is only to simplify the argument below anyway). For each $\lambda \in W$ we get, by the universal property of G_r an endomorphism $\rho_\lambda : G_r \longrightarrow G_r$ taking α_r to $\lambda\alpha_r$. By functoriality it acts on $t(G_r)$ and commutes with $G_r^j \longrightarrow \underline{\underline{S}}(t(G_r))^j$. Using the descriptions of G_r and $t(G_r)$ we see that they split up into sums of pieces where ρ_λ for all λ acts by multiplication by λ^{σ^k} for some k. Because the automorphisme σ^k, $k \in \mathbb{Z}$, are all distinct this sum is necessarily direct and we may restrict our attention to one piece say where ρ_λ acts by λ^{σ^f}. The part of G_r^j with this property will then be

$W\underline{F}^f\varepsilon$ if $j \geq r$ and $f \geq 0$ and $W\underline{V}^f\varphi$ if $f < 0$ etc. The part of $\underline{\underline{S}}(t(G_r))$ will be generated over W by $F^f(\beta_i \otimes \alpha_r)$ $(V^{-f}(\beta_i \otimes \alpha_r))$ if $i < \min(j,r)$; $F^{f-1}(\beta_i \otimes \alpha_r)(V^{-f-1}(\beta_i \otimes \alpha_r))$ if $r \leq i < j$; $F^{f+1}(\beta_i \otimes \alpha_r)(V^{-f+1}(\beta_i \otimes \alpha_r))$ if $j \leq i < r$; $F^f(\beta_i \otimes \alpha_r)(V^{-f}(\beta_i \otimes \alpha_r))$ if $i \geq \max(j,r)$ and

$$F^f d(\beta_i \otimes \alpha_r)(dV^{-f}(\beta_i \otimes \alpha_r)) \quad \text{if } i < \min(j-1,r) ;$$

$$F^{f-1} d(\beta_i \otimes \alpha_r)(dV^{-f-1}(\beta_i \otimes \alpha_r)) \quad \text{if } r \leq i < j-1 ;$$

$$F^{f+1} d(\beta_i \otimes \alpha_r)(dV^{-f+1}(\beta_i \otimes \alpha_r)) \quad \text{if } j-1 \leq i < r$$

$F^f d(\beta_i \otimes \alpha_r)(dV^{-f}(\beta_i \otimes \alpha_r))$ if $i \geq \max(j-1,r)$ when $f \geq 0$ $(f < 0)$. Let us call the λ^{σ^f}-eigenspace of G_r^j Y_f and the one of $\underline{\underline{S}}(t(G_r))^j$ X_f so that we have a map $Y_f \longrightarrow X_f$ of complexes and we want to show that it is a quasi-isomorphism. We distinguish between $f \geq 0$ and $f < 0$. Case I : $f \geq 0$: let us first show that $H^1(X_f) = 0$. When $f > 0$ already δ is surjective (recall that δ is the differential of $t(G_r)$ as R-complex). This is true also if $j \leq r$ so we may assume $r < j$. Then if we put d' the other differential of $t(G_r)(\varphi^j)(\varphi^j(n) = -\delta_{jn})$ we have that d' is surjective except that $d(\beta_{j-1} \otimes \alpha_r)$ is not hit (i.e. $d(\beta_j \otimes \alpha_r)$ generates the cokernel of d') but it is hit by δ so that we first find an α such that $d(\beta_{j-1} \otimes \alpha_r) \equiv \delta\alpha$ mod Coker d' and then hit $d(\beta_{j-1} \otimes \alpha_r) - \delta\alpha$ by d'.

The next step is to observe that projection of X_f^0 onto the $F^f(\beta_f \otimes \alpha_r)$-factor is injective on Z^0X_f. This first shows that $Y_f \longrightarrow Z^0X_f$ is injective as $\underline{F}^f\varepsilon$ or $\underline{F}^{f+1}\varepsilon$ maps to $F^f(\beta_m \otimes \alpha_r)$ resp. $F^{f+1}(\beta_m \otimes \alpha_r)$ which is non-zero, then that in order to show that $Y_f \longrightarrow Z^0X_f$ is surjective it suffices to show that the image of Y_f by the projection onto the $F^f(\beta_m \otimes \alpha_r)$-factor contains the image Z^0X_f. However, Y_f maps surjectively to this factor. Case II :

$f<0$. (Put $n=-f$: Arguments very similar to the preceding ones show that $Y_f \longrightarrow Z^0X_f$ is injective and that $H^1(X_f)=0$ and that we may again project to the $V^{-n}(p_m \otimes \alpha_r)$-factor.

Let us first assume that $j \leq r$. Then (4.5.2) shows that Y_f maps to $p^{(n-m)h+r-j}WV^h(\beta_m \otimes \alpha_r)$ so we have to show that the $V^h(\beta_m \otimes \alpha_r)$-factor of any element of Z^0X_f is divisible by $p^{(n-m)h+r-j}$. We have γ_i $m \leq i \leq n$ $t(G_r)$ where γ_i has R-module degree i, ρ_λ acts on it by λ^{σ^f} and $d'\gamma_i = \delta\gamma_{i+1}$ and we then want to show that γ_m is divisible by $p^{(n-m)h+r-j}$.

Put

$$\gamma_i = \begin{cases} a_i V^h(\beta_i \otimes \alpha_r) & i<j-1 \text{ or } i \geq r \\ a_i V^{h+1}(\beta_i \otimes \alpha_r) & \text{if not .} \end{cases}$$

Then $d'\gamma_i = \delta\gamma_{i+1}$ for $m \leq i < n$ is equivalent to

$$\begin{aligned} a_i &= p^h a_{i+1} && i<j-1 \text{ or } i \geq r \\ a_i^{\sigma^{-1}} &= p^{h+1} a_{i+1} && i=j-1 \\ a_i &= p^{h+1} a_{i+1} && j-1<i<r-1 \\ a_i &= p^h a_{i+1}^{\sigma^{-1}} && i=r-1 \text{ .} \end{aligned}$$

From this we get $a_m = p^{(n-m)h+r-j}a_n$ so that indeed γ_m is divisible by $p^{(n-m)h+r-j}$. The case $j>r$ is very similar and left to the reader.

Any F-gauge structure may be regarded as a graded $\tilde{R} := W[\tilde{F},\tilde{V}]/(\tilde{F}\tilde{V}-p)$-module where $\deg \tilde{F} = 1$ and $\deg \tilde{V} = -1$. We also get rings $k[\tilde{F}] := \tilde{R}/(\tilde{V})$, $k[\tilde{V}] := \tilde{R}/(\tilde{F})$ $k := \tilde{R}/(\tilde{F},\tilde{V})$.

Definition 4.6. i) *Let* $M \in D(F\text{-g-str})$. *Then there is a commutative, up to sign, diagram of distinguished triangles*

$$
(4.6.1)\qquad
\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & M & \xrightarrow{\tilde F} & M(1) & \rightarrow & M/\tilde F(1) & \rightarrow \\
 & \downarrow{\scriptstyle \tilde V} & & \downarrow{\scriptstyle \tilde V} & & \downarrow{\scriptstyle \tilde V} & \\
 & M(-1) & \xrightarrow{\tilde F} & M & \rightarrow & M/\tilde F & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & M/\tilde V(-1) & \xrightarrow{\tilde F} & M/\tilde V & \rightarrow & M/(\tilde V,\tilde F) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

ii) *Let* $M \in D(F\text{-g-str})$. *Then there is a distinguished triangle*

$$(4.6.2)\qquad \rightarrow M/\tilde F \xrightarrow{\tilde V} W/p \otimes^{L}_{W} M \rightarrow M/\tilde V \rightarrow$$

iii) *Let* $N \in D(R)$. *Then* (4.6.1) *for* $M = \underset{=}{S}(N)$ *is isomorphic to* (3.4.1).

Proof : (4.6.1) follows from the exact sequences

$$0 \rightarrow \tilde R(-1) \xrightarrow{\tilde V} \tilde R \rightarrow k[\tilde F] \rightarrow 0$$

$$0 \rightarrow \tilde R(1) \xrightarrow{\tilde F} \tilde R \rightarrow k[\tilde V] \rightarrow 0$$

$$0 \rightarrow R \xrightarrow{(\tilde F,-\tilde V)} \tilde R(1) \oplus \tilde R(-1) \xrightarrow{(\tilde V,\tilde F)} \tilde R \rightarrow k \rightarrow 0$$

ii) is clear and iii) follows from the proof of (3.4).

Proposition 4.7. i) $L\hat t$ *has finite amplitude on* $D(F\text{-g-str-}I)$, I *finite, so extends to all of* $D(\text{g-str-}I)$.

ii) $\underset{=}{S}$ *and* $L\hat t$ *give inverse equivalences between complete* R-*complexes of level* I *and complete* F-*complexes of* F-*gauge-structures of level* I . *They take bounded complexes to bounded complexes.*

Indeed, by (0:4) $L\hat t$ has finite amplitude iff $R_1 \otimes^{L}_{R} L\hat t$ has. Now as $R_1 \otimes^{L}_{R} (-) = \underset{=}{S}(-)/(\tilde F,\tilde V)$ by (3.6) and as $\underset{=}{S}$ is an inverse of $L\hat t$ we get

$R_1 \otimes^L_R L\hat{t}(-) = (-)/(\tilde{F},\tilde{V})$ and $(-)/(\tilde{F},\tilde{V})$ has finite amplitude. Thus i) follows and ii) will follow if we can show that a complete M $D(F\text{-g-str-}I)$ is bounded if $M/(\tilde{F},\tilde{V})$ is. However, the grading of $M/\tilde{F}$ and $M/\tilde{V}$ have amplitude $]-\infty,n]$ resp. $[m,\infty[$ if $I=[m,n]$ so (3.6.1) shows that $M/\tilde{F}$ and $M/\tilde{V}$ are bounded if $M/(\tilde{F},\tilde{V})$ are. Then (4.6.2) shows that $W/p \otimes^L_W M$ is bounded but Nakayama's lemma shows that a complete complex is bounded if $W/p \otimes^L_W M$ is .

5. We will now begin to impose finiteness conditions on F-gauge structures.

Definition 5.1. i) *An* $M \in D(F\text{-g-str-}I)$, I *finite, is coherent iff* M *is complete and* $W/p \otimes^L_W M$ *is a coherent* k-*complex i.e. with finite dimensional cohomology spaces.*

ii) *An* F-*gauge structure of level* I *is coherent iff it is coherent as a complex of* F-*gauge structures of level* I .

Proposition 5.2. i) *Let* $M \in D^-(F\text{-g-str-}I)$. *Then* M *is coherent iff* $\forall i \in I$ $j \in \mathbb{Z}$ $H^j(M)^i$ *is a finitely generated* W-*module.*

ii) *A complete* $M \in D(F\text{-g-str-}I)$ *is coherent iff* $M/(\tilde{F},\tilde{V})$ *is a coherent* k-*complex.*

Proof : It is clear that if $M \in F\text{-g-str-}I$ and M^i is finitely generated for all i then M is coherent. This gives one direction. For the other we may apply descending induction so that we may assume that $H^r(M)=0$ for $j<r$ and then it suffices to show that $H^j(M)^i$ is of finite type for all i . The completeness of M implies that $H^j(M)^i$ is a p-adically complete W-module and $W/p \otimes^L_W M$ coherent implies that $H^j(M)^i/p$ is finite dimensional. This together implies by Nakayama's lemma that $H^r(M)^i$ is of finite type. As for ii) (4.6.2) shows that if $M/\tilde{F}$ and $M/\tilde{V}$ are coherent k-complexes then so is $W/p \otimes^L_W M$. However, if $I=[m,n]$ then $M/\tilde{F}$ and $M/\tilde{V}$ are zero in module degrees above n resp. below and (4.6.1) and the coherence of $M/(\tilde{F},\tilde{V})$ show that modulo coherent complexes $M/\tilde{F}$ resp. $M/\tilde{V}$

is isomorphic to its module shift one step to the right resp. to the left which clearly implies that they are coherent.

We will denote by $D_c(F\text{-g-str-}I)$ the category of coherent complexes. Putting together (4.7 ii) and (5.2 ii) we get

<u>Theorem</u> 5.3. $\underline{\underline{S}}$ *and* $L\hat{t}$ *give inverse equivalences between* $D_c(R\text{-}I)$ *and* $D_c(F\text{-g-str-}I)$ *and* $D_c^b(R\text{-}I)$ *and* $D_c^b(F\text{-g-str-}I)$.

Of course $\underline{\underline{S}}$ and $L\hat{t}$ do not preserve the standard t-structures. Let $(G^{\leq 0}\text{-}I, G^{\geq 0}\text{-}I)$ denote the t-structure induced on $D_c^b(R\text{-}I)$ by $(G^{\leq 0}, G^{\geq 0})$.

<u>Proposition</u> 5.4. $\underline{\underline{S}}$ *takes* $(G^{\leq 0}\text{-}I, G^{\geq 0}\text{-}I)$ *to the standard* t-*structure*.

It clearly suffices to show that if $M \in D_c^b(R\text{-}I)$ and $\underline{\underline{S}}(M) \in F\text{-g-str-}I$ then $M \in G$ or by (1.2) that $\underline{\underline{s}}(M) \in W\text{-mod}$ and that $R_1 \otimes_R^L M$ has amplitude $[-2,0]$. However $\underline{\underline{s}}(M) = \underline{\underline{S}}(M)^{\infty}$ so clearly $\underline{\underline{s}}(M) \in W\text{-mod}$ and (3.4) shows that $R_1 \otimes_R^L M$ has amplitude $[-2,0]$.

It is clear that $L\hat{t}(M)$ is not a very convenient description of the R-complex associated to M. Our purpose is now to give a more explicit description in some cases. Here comes a point where we will prefer to work with the slight variant of F-gauge structures of level I mentioned before i.e. where we require an extension of the action of $W_\sigma[\underline{F},\underline{V}]$ to its p-adic completion and work with $\hat{R}$ instead of R. Let us call such objects $\hat{F}$-gauge-structures. We will denote by $t'(-)$ the functor from $\hat{F}$-gauge structures to complexes of $\hat{R}$-modules corresponding to $t(-)$. As coherent F-gauge structures are p-adically complete they may be regarded as $\hat{F}$-gauge structures. More generally we can take the p-adic completion of any F-gauge structure of finite level.

In particular the completions of the G_r we will denote $\hat{G}_r$. It is clear that they form a set of projective generators for $\hat{F}\text{-g-str-}I$.

Lemma 5.5. *The* $\hat{G}_r$ *are noetherian* $\hat{F}$*-gauge structures of level* I .

Indeed, they are finite type modules over the p-adic completion of $W_\sigma[\underline{F},\underline{V}]$ which is well known to be noetherian.

Theorem 5.6. *Let* M *be a coherent* F*-gauge structure of level* I *such that* $L\hat{t}(M) \in \Delta\text{-}I$ $(:= \Delta \cap D(R\text{-}I))$. *Then* $L\hat{t}(M) = t'(M)$.

Proof : As M is coherent we can by lemma 5.5 find a resolution $H^\bullet \longrightarrow M$ of M such that H^i is a finite sum of $\hat{G}_r$:s . Then $L\hat{t}(M)$ is the completion of $t'(H^\bullet)$. I claim that $t'(H^\bullet)$ already is complete. For this it suffices to show that $t'(\hat{G}_r)$ is complete. The analogue of (4.2) gives that $t'(\hat{G}_r) = \bigoplus_{m \leq i < n} \hat{R}(\beta_i \otimes \alpha_r) + W\{F,F^{-1}\}(\beta_n \otimes \alpha_r)$ where $W\{F,F^{-1}\}$ is the p-adic completion of $W[F,F^{-1}]$. As $\hat{R}$ and $W\{F,F^{-1}\}$ are complete as R-complexes we are through. Hence $L\hat{t}(M) = t'(H^\bullet)$ and we get a well defined morphism $L\hat{t}(M) = t'(H^\bullet) \longrightarrow t'(M)$. By definition $t'(H^\bullet)$ is the simple complex associated to the double complex obtained by applying $t'(-)$ pointwise to $H^\bullet$. We will denote this double complex $t''(H^\bullet)$ and we have a mapping of double complexes $t''(H^\bullet) \longrightarrow t'(M)$ of $\hat{R}$-modules where the simple complex $t'(M)$ is regarded as a double complex in the usual way. We will now consider the morphism of spectral sequences associated to this morphism of double complexes. We have $E_1^{j,i} = (t'(H^j)^i, d')$ (resp. $(t'(M)^i, 0)$ if $j = 0$ and zero if not) where $(\ldots \xrightarrow{d'} H^{j-1} \xrightarrow{d'} H^j \xrightarrow{d'} H^{j+1} \xrightarrow{d'} \ldots)$. Note first that by the definition of $t'(-)$ $t'(H^j)^i$ resp. $t'(M)^i$ are concentrated in degrees $-i$ and $-i+1$. As the two spectral sequences are spectral sequences of R-modules this shows that $d_r = 0$ if $r \geq 3$. Furthermore, $t'(-)$ is a left adjoint so right exact which shows that $E_2^{0,i}(H^\bullet) \longrightarrow E_2^{0,i}(M)$ is an isomorphism for each i and as $d_r = 0$ if $r \geq 3$ $E_\infty^{0,i}(H^\bullet) \longrightarrow E_\infty^{0,i}(M)$ is still an isomorphism. Now the spectral sequences converge to $H^i(L\hat{t}(M))$ resp. $H^i(t'(M))$ so we get a commutative diagram

$$\begin{array}{ccc} E_\infty^{0,i}(H^\cdot) & \xrightarrow{\sim} & E_\infty^{0,i}(M) \\ \downarrow & & \downarrow\wr \\ H^i(L\hat{t}(M)) & \longrightarrow & H^i(t'(M)) \end{array} .$$

This shows that $L\hat{t}(M) \longrightarrow t'(M)$ induces a split surjection on cohomology. Furthermore, the kernel of $H^i(L\hat{t}(M)) \longrightarrow H^i(t'(M))$ being an extension of subquotients of the $E_1^{j,k}(H^\cdot)$ for $k<i$ is concentrated in module degrees $\leq -i$ (so far we have not used that $L\hat{t}(M) \in \Delta$). I claim that this implies that $t'(M) \in \Delta$, that $L\hat{t}(M) \longrightarrow t'(M)$ is surjective in Δ and that its kernel is Hodge-Witt. Indeed, construct a distinguished triangle $\longrightarrow K \longrightarrow L\hat{t}(M) \longrightarrow t'(M) \longrightarrow$ and consider the long exact cohomology sequence $\ldots \longrightarrow H^i(K) \longrightarrow H^i(L\hat{t}(M) \longrightarrow H^i(t'(M)) \longrightarrow \ldots$ The fact that $H^i(L\hat{t}(M)) \longrightarrow H^i(t'(M))$ is a split surjection implies that $H^i(L\hat{t}(M)) = H^i(t'(M)) \oplus H^i(K)$. This gives first that $H^i(t'(M))$ and $H^i(K)$ are coherent being direct factors of something coherent, hence that $t'(M)$ and K are coherent and secondly that $H^i(t'(M))$ and $H^i(K)$ are generated by its degree $-i$ part being images of something that is, which by definition means that $K, t'(M) \in \Delta$. Finally $H^i(K)$ is concentrated in degrees $\leq -i$ and is generated by its degree $-i$ part and therefore it is a shifted Dieudonné module so K is Hodge-Witt. As $K, t'(M) \in \Delta$ the distinguished triangle $\longrightarrow K \longrightarrow L\hat{t}(M) \longrightarrow t'(M) \longrightarrow$ is in fact a short exact sequence in Δ.

We have to any $N \in \Delta\text{-I} \cap G\text{-I}$ associated two objects $a(N) := t'(\underline{\underline{S}}(N))$ and $b(N)$ in $G\text{-I} \cap \Delta\text{-I}$ such that $b(N)$ is Hodge-Witt and we have obtained a functorial short exact sequence

$$0 \longrightarrow b(N) \longrightarrow N \longrightarrow a(N) \longrightarrow 0 .$$

We want to show that $b(N)$ is 0. One case is clear namely when N contains no Hodge-Witt subobjects. Suppose that we also know it when N is Hodge-Witt. Consider the exact sequence in Δ

$$0 \longrightarrow HW(N) \longrightarrow N \longrightarrow N/HW(N) \longrightarrow 0 .$$

By definition of G an $N \in \Delta$ is in G iff it has no non-trivial $\underline{\underline{s}}$-1-torsion quotient. Hence any quotient in Δ of N is also in G. On the hand, any Hodge-Witt diagonal complex is also in G so both $HW(N)$ and $N/HW(N)$ are in G. Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & HW(N) \arrow[r, "\sim"] \arrow[d] & a(HW(N)) \arrow[d] & \\
0 \arrow[r] b(N) \arrow[r] & N \arrow[r] \arrow[d] & a(N) \arrow[r] \arrow[d] & 0 \\
 & N/HW(N) \arrow[r, "\sim"] \arrow[d] & a(N/HW(N)) & \\
 & 0 & &
\end{tikzcd}$$

The fact that $N/HW(N) \xrightarrow{\sim} a(N/HW(N))$ implies that $b(N) \subseteq HW(N)$. If we can show that $a(HW(N)) \longrightarrow a(N)$ is a monomorphism we immediately get $b(N) = 0$. Let L be the kernel of $a(HW(N)) \longrightarrow a(N)$. As $a(HW(N)) = HW(N)$ is Hodge-Witt L is Hodge-Witt and by (I:1.5.1) $H^i(L) \longrightarrow H^i(a(HW(N)))$ is a monomorphism. However, we have seen that $H^i(N) \longrightarrow H^i(a(N))$ is split and the splitting is clearly functorial so $H^i(a(HW(N))) \longrightarrow H^i(a(N))$ is a direct factor of $H^i(HW(N)) \longrightarrow H^i(N)$ which again by (I:2.2.2) is injective. Therefore $H^i(L) = 0$ and so $L = 0$.

We are hence left with the case when $N \in \Delta_{HW}$ and we want to show that $N = t'(\underline{\underline{S}}(N))$ or properly speaking $N = t'(H^0(\underline{\underline{S}}(N)))$. However, by (0:5.1) we may assume that $N = \bigoplus_i H^i(N)[-i]$ and then $\underline{\underline{S}}(N)$ is an F-gauge structure and not just quasi-isomorphic to one. Hence we have a morphism of complexes $L\hat{t}(\underline{\underline{S}}(N)) = t'(H^\bullet) \longrightarrow t'(\underline{\underline{S}}(N))$ where $H^\bullet \longrightarrow \underline{\underline{S}}(N)$ is a finite type projec-

tive resolution of $\underline{\underline{S}}(N)$ as before. The composite $L\hat{t}(\underline{\underline{S}}(N)) \longrightarrow t'(\underline{\underline{S}}(N)) \xrightarrow{\text{adj}} N$ is an isomorphism. However, we have already seen that $L\hat{t}(\underline{\underline{S}}(N)) \longrightarrow t'(\underline{\underline{S}}(N))$ is a split epimorphism on cohomology so it is an isomorphism.

Corollary 5.6.1. *Let* M *be a coherent* F-*gauge structure. Then* $t'(M) = \widetilde{H}^0(L\hat{t}(M))$.

Proof : We saw in the proof of the theorem that $L\hat{t}(M) \longrightarrow t'(M)$ induced a split surjection on cohomology and that by construction $t'(M)^i$ is concentrated in degree $-i$ and $-i+1$. As $L\hat{t}(M) \in G \subseteq \Delta^{\leq 0}$ $H^i(L\hat{t}(M))$ is generated by its part of degree less than or equal to $-i$ and so is $H^i(t'(M))$ being a direct factor which also shows that $H^i(t'(M))$ is coherent. As it is also concentrated in degrees $-i$ and $-i+1$ it is generated by its degree $-i$ part so $t'(M)$ is a diagonal complex. Consider the diagram in $D(R)$

$$\begin{tikzcd} L\hat{t}M \arrow[r] \arrow[d, "\psi"'] & \widetilde{H}^0(L\hat{t}(M)) \arrow[d, "\varphi"] \arrow[dl, dashed] \\ t'(M) \arrow[r] & t'(H^0(\underline{\underline{S}}(\widetilde{H}^0(L\hat{t}(M)))) \end{tikzcd}$$

By the theorem φ is an isomorphism. As $t'(M)$ is in Δ ψ factors through $\widetilde{H}^0(L\hat{t}(M))$ and as φ is an isomorphism $\widetilde{H}^0(L\hat{t}(M)) \longrightarrow t'(M)$ is a split monomorphism as is a fortiori the map induced on cohomology. However, the map on cohomology induced by ψ is a split surjection so $\widetilde{H}^0(L\hat{t}(M)) \longrightarrow t'(M)$ is a quasi-isomorphism.

Diagonal complexes revisited

1. We can now obtain more information about diagonal complexes by using the results of II.

<u>Proposition</u> 1.1. *The functor* $\underline{\underline{S}}((-)[1])$ *gives an equivalence from the category of* $\underline{\underline{s}}$*-acyclic diagonal complexes to the category of coherent* F*-gauge structures* M *with* $M^{\infty} = 0$.

<u>Proof</u> : An $\underline{\underline{s}}$-acyclic diagonal complex N is $\underline{\underline{s}}$-1-torsion so by definition $N[1] \in G$ and $\underline{\underline{S}}(N[1]) \in$ F-g-str . Furthermore, $\underline{\underline{S}}(N[1])^{\infty} = \underline{\underline{s}}(N[1]) = 0$. Conversely, if M is a coherent F-gauge structure with $M^{\infty} = 0$ then $\underline{\underline{s}}(L\hat{t}(M)) = 0$ so by (I:4.2 iv) $\widetilde{H}^{i}(L\hat{t}(M))$ is $\underline{\underline{s}}$-acyclic for all and in particular $\underline{\underline{s}}$-1-torsion. This implies that $\widetilde{H}^{i}(L\hat{t}(M)) = 0$ if $i \neq -1$ so $L\hat{t}(M)[-1]$ is an $\underline{\underline{s}}$-acyclic diagonal complex.

<u>Definition</u> 1.2. *Let* $I = [m,n]$ *be a finite interval and* J *a subset of* $[m,n-1]$. *Let* $M(I,J)$ *be the* F*-gauge structure with* $M(I,J)^{i} = k$ *if* $i \in I$, 0 *if not,* $\widetilde{F} : M(I,J)^{i} \longrightarrow M(I,J)^{i+1}$ *is the identity if* $i \in J$ *and* 0 *if not and* $\widetilde{V} : M(I,J)^{i} \longrightarrow M(I,J)^{i+1}$ *is the identity if* $i \in I \ J$ *and* 0 *if not. We will denote by* $M(I,J)$ *also the associated* $\underline{\underline{s}}$*-acyclic diagonal complex.*

It is convenient to represent $M(I,J)$ by a directed graph Γ as follows. The vertex set of Γ is I and there is an arrow from i to i+1 if $\widetilde{F} : M(I,J)^{i} \longrightarrow M(I,J)^{i+1}$ is the identity and an arrow from i+1 to i $\widetilde{V} : M(I,J)^{i+1} \longrightarrow M(I,J)^{i}$ is the identity. Thus $M(\{0,1,2\},\{1\})$ will have $\overset{0}{\bullet} \longleftarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ as graph.

<u>Proposition</u> 1.3. *The* $M(I,J)$ *are precisely the indecomposable* $\underline{\underline{s}}$*-acyclic diagonal complexes killed by* p *if* $H=1$.

Indeed, let N be the F-gauge structure associated to an indecomposable $\underline{\underline{s}}$-acyclic diagonal complex killed by p and let $I=[m,n]$ be its level. By induction we may assume the result true for level $I':=[m+1,n]$. Let N' denote the F-gauge structure with $N'^i=N^i$ if $i\neq m$ and $N'^m=0$ and the same $\tilde{F}:s$ and $\tilde{V}:s$ as N except between N'^m and N'^{m+1} . By induction N' is a sum of $M(I'',J):s$ with $I''\subseteq I'$ and in fact necessarily have $m+1\in I''$ because a factor $M(I'',J)$ in N' with $m+1\notin I''$ would give a non-trivial factor of N .

Let us admit for the moment that for any two pairs (I_1,J_1) and (I_2,J_2) with $I_1,I_2\subseteq I'$ $m-1\in I_1,I_2$, $J_1\subseteq I_1\cap[m-1,n-1]$ and $J_2\subseteq I_2\cap[m+1,n-1]$ there is either a morphism $M(I_1,J_1)\longrightarrow M(I_1,J_2)$ or one $M(I_2,J_2)\longrightarrow M(I_1,J_1)$ inducing the identity in degree $m+1$.

Suppose that $\tilde{F}:N^m\longrightarrow N^{m+1}$ is non-zero, pick some $k\subseteq N^m$ such that $\tilde{F}_{|k}\neq 0$ and fix some decomposition $\oplus N_i$ of N' into indecomposables. Using the admitted resulted it is clear we can find an automorphism ψ of N' which is a product of "elementary matrices" of the form $id+\varphi$, $\varphi:N_i\longrightarrow N_j$, $i\neq j$, such that the image of k by $\tilde{F}$ lies in one of the factors N_{i_0} of the decomposition $\psi(\oplus N_i)$ of N'. As $\tilde{F}:k\longrightarrow N_{i_0}^{m+1}$ is an isomorphism we can find a complement T of k in N^m such that the composite $T\hookrightarrow N^m\xrightarrow{\tilde{F}}N^{m+1}\longrightarrow N_{i_0}^{m+1}$ is zero and hence $N'':=(\ldots\ 0\rightleftarrows k\xrightarrow{\tilde{F}}N_{i_0}^{m+1}\underset{\tilde{V}}{\overset{\tilde{F}}{\rightleftarrows}}N_{i_0}^{m+2}\rightleftarrows\ldots)$ is a non-zero direct factor of N and so equals N . Clearly N'' is of the form $M(\bar{I},J)$. Similarly, if $\tilde{V}$ is non-zero. If both $\tilde{V}$ and $\tilde{F}$ are zero then $N=N^m\oplus N'$ so either $N=N^m$ in which case $N=k(-m)$ or $N=N'$ where the result has been assumed by induction.

It remains to show the result we have so far admitted as the $M(I,J)$ are clearly indecomposable. Let r be the largest integer $\leq n$ such that

if $m+1 \leq i \leq r$ then $i \in J_1$ iff $i \in J_2$. Suppose $r = n$. By symmetry we may assume $I_1 \subseteq I_2$ so that $J_1 = I_1 \cap J_2$. If $I_1 = I_2$ then $M(I_1,J_1) = M(I_2,J_2)$ and there is nothing to prove. If not let $I_1 = [m+1,s]$.

Then the graphs are like the following

$$M(I_1,J_1) : \xrightarrow{m+1} \rightarrowtail\!\leftarrow\!\bullet \; \cdots \; \xrightarrow{s}$$

$$M(I_2,J_2) : \quad \text{—"—} \qquad \leftarrow\!\bullet \; \cdots$$

or

$$\text{—"—} \qquad \bullet\!\!\rightarrow \; \cdots$$

In the first case $M(I_1,J_1) \hookrightarrow M(I_2,J_2)$ and in the second $M(I_2,J_2) \twoheadrightarrow M(I_1,J_1)$ and both morphisms induce the identity in degree $m+1$.

In the general case the graphs of $M(I_1,J_1)$ and $M(I_2,J_2)$ are the same up to degree $r+1$ and then one continues with $\xrightarrow{r+1 \quad r+2}$ and the other $\xleftarrow{r+1 \quad r+2}$. In the $\bullet\!\!\rightarrow$-case there is an epimorphism to $M([m+1,r+1], J_1 \cap [m+1,r]) = M([m+1,r+1], J_2 \cap [m+1,r])$ and in the $\leftarrow\!\bullet$-case we have $M([m+1,r+1], J_1 \cap [m+1,r])$ as subobject in both cases inducing the identity in degree $m+1$. Composing these two morphisms we get what we want.

Diagonal dominoes

Definition 2.1. *A diagonal domino is an* $\underline{\underline{s}}$*-torsion diagonal complex without finite torsion.*

It is clear that a subobject of a diagonal domino and an extension of diagonal dominoes is again a diagonal domino. Also any $\underline{\underline{s}}$-1-torsion object contains no finite torsion as its finite torsion would be at the same time $\underline{\underline{s}}$-1-torsion and Hodge-Witt and so zero by "survie du coeur". In particular $\underline{\underline{s}}$-acyclic objects are diagonal dominoes. Finally, by (I:2.2.2), any domino is a diagonal domino.

Lemma 2.2. i) $(H=1)$ *Let* $M \in \Delta$ *be* $\underline{\underline{s}}$-*torsion. Then* M *has no non-zero* $\underline{\underline{s}}$-0-*torsion* ($\underline{\underline{s}}$-1-*torsion*) *subobject (resp. quotient object) iff* $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^k,M)=0$ $(\mathrm{Hom}_\Delta(M,\underline{\underline{U}}_0^k)=0)$ *for all* k.

ii) *If* M *is a diagonal domino then so is* $\widetilde{D}^0(M)$ *and* $M=\widetilde{D}^0(\widetilde{D}^0(M))$. *Also* M *is* $\underline{\underline{s}}$-0-*torsion iff* $\widetilde{D}^0(M)$ *is* $\underline{\underline{s}}$-1-*torsion.*

iii) *Let* M *be an* $\underline{\underline{s}}$-0-*torsion* ($\underline{\underline{s}}$-1-*torsion*) *diagonal domino and let* $\varphi:\mathbb{Z}\longrightarrow\mathbb{N}$ *then* $M(\varphi)$ $(M(-\varphi))$ *is an* $\underline{\underline{s}}$-0-*torsion (resp.* $\underline{\underline{s}}$-1-*torsion) diagonal domino.*

Suppose M $\underline{\underline{s}}$-torsion with no non-zero $\underline{\underline{s}}$-0-torsion subobject. As the image of any $\underline{\underline{U}}_0^k \longrightarrow M$ is $\underline{\underline{s}}$-0-torsion by (I:4.2) $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^k,M)=0$. Conversely, if $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^k,M)=0$ then this is also true for any subobject so it will suffice to prove that $H^0(\underline{\underline{s}}(M))=0$. As $H^0(\underline{\underline{s}}(M))$ is torsion by assumption it will suffice to prove that $H^{-1}(k \otimes_W^L \underline{\underline{s}}(M))=0$. By (0:4.3) $H^i(R_1 \otimes_R^L M)^j=0$ if $i+j=-1$ so the spectral sequence

$$H^i(R_1 \otimes_R^L M)^j \Longrightarrow H^{i+j}(k \otimes_W^L \underline{\underline{s}}(M))$$

gives $H^{-1}(k \otimes_W^L \underline{\underline{s}}(M))=0$. The second part of i) is proved in a similar way. Let M be a diagonal domino. Then $\widetilde{D}^0(M)$ is without finite torsion by (I:1.11) and $D(M)=\widetilde{D}^0(M)$ by (I:1.8). Hence $\widetilde{D}^0(M)$ is $\underline{\underline{s}}$-torsion by the formula $\underline{\underline{s}}(D(M))=\mathrm{RHom}_W(\underline{\underline{s}}(M),W)$ and $M=\widetilde{D}^0(\widetilde{D}^0(M))$ by (I:1.11). Finally $\underline{\underline{s}}(D(M))=\mathrm{RHom}_W(\underline{\underline{s}}(M),W)$ shows that $\widetilde{D}^0(M)$ is $\underline{\underline{s}}$-0-torsion iff M is $\underline{\underline{s}}$-1-torsion.

To prove iii) we may by ii), the fact that $M(\varphi)$ is clearly without finite torsion and $D((-)(\varphi))=D(-)(-\varphi)$, assume that we are in the resp. case and by the formula $M(\varphi)(\psi)=M(\varphi+\psi)$ that $\varphi(n)=\delta_{in}$ for some i so that $\underline{\underline{s}}(M(\varphi))=\underline{\underline{S}}(M)^i$. However by definition $M[1]\in G$ so by (II:4.4) $\underline{\underline{s}}(M(\varphi))$ is concentrated in degree 1.

Let us introduce the notation $m_i(-)$ for $(-)(\varphi)$ with $(n)=-i$ for all n. Hence $m_i(\underline{\underline{U}}_0^j)=\underline{\underline{U}}_{-i}^j$ and m_i takes $\underline{\underline{s}}$-0-torsion ($\underline{\underline{s}}$-1-torsion)

diagonal dominoes to $\underline{\underline{s}}$-0-torsion ($\underline{\underline{s}}$-1-torsion) diagonal dominoes if $i \leq 0$ ($i \geq 0$). Note also that $m_i(m_j(-)) = m_{i+j}(-)$ and $m_0 = \mathrm{id}$.

Theorem 2.3. i) *Any diagonal domino admits an exhaustive, separated and finite filtration by admissible subobjects*

$$M \supseteq \ldots \supseteq W^{s-1/2} \supseteq W^s \supseteq W^{s+1/2} \supseteq \ldots \supseteq 0 \qquad 2s \in \mathbb{Z}$$

such that

a) *if* $s \in \mathbb{Z}$ $m_s(W^s/W^{s+1/2})$ *is* $\underline{\underline{s}}$*-acyclic*

b) *if* $s = r-1/2$, $r \in \mathbb{Z}$, *then* $\forall k \in \mathbb{Z}$

$$\mathrm{Hom}_{\Delta}(\underline{\underline{U}}^k_0, m_r(W^s/W^{s+1/2})) = \mathrm{Hom}_{\Delta}(m_r(W^s/W^{s+1/2}), \underline{\underline{U}}^k_{-1}) = 0 \ .$$

ii) *The filtration is determined by* a) *and* b), W^s *is an idempotent radical and* $m_r(W^s(-)) = W^{s-r}(m_r(-))$.

iii) $\tilde{D}^0(W^s(-)) = \tilde{D}^0(-)/W^{-s+1/2}\tilde{D}^0(-)$.

This filtration will be referred to as the type filtration or the filtration by type.

Proof : Let M be a diagonal domino and let F^1, F^2, G^1 and G^2 have the meanings of (I:4.5). As $F^2/F^1 = G^2/G^1$ is Mazur-Ogus $H^1(\underline{\underline{s}}(F^2/F^1)) = 0$ and (I:4.5.3) and the fact that M is $\underline{\underline{s}}$-torsion show that $H^0(\underline{\underline{s}}(F^2/F^1))$ so F^2/F^1 is $\underline{\underline{s}}$-acyclic but by the definition of the Mazur-Ogus property this gives $F^2/F^1 = G^2/G^1 = 0$ so $F^2 = F^1$ and $G^2 = G^1$. Hence $0 \subseteq G^1 \subseteq F^1 \subseteq M$, F^1 is the largest $\underline{\underline{s}}$-0-torsion subobject, M/G^1 the largest $\underline{\underline{s}}$-1-torsion quotient of M , by (I:4.5) F^1/G^1 is $\underline{\underline{s}}$-acyclic and by (I:4.5.3) M/F^1 is $\underline{\underline{s}}$-1-torsion so a diagonal domino. Thus G^1 and F^1 are admissible subobjects and they are also clearly idempotent radicals. I claim that if $r < 0$ then $m_rF^1 \subseteq G^1(m_rM)$ and if $r > 0$ then $F^1(m_rM) \leq m_rG^1$. Indeed, consider the first inclusion. By lemma 2.2 m_rF^1 is $\underline{\underline{s}}$-0-torsion so $m_rF^1 \longrightarrow F^1(m_rM)$. Now consider $m_rF^1 \longrightarrow F^1(m_rM)/G^1(m_rM)$. The left hand side

is $\underline{\underline{s}}$-0-torsion and the right hand side is $\underline{\underline{s}}$-1-torsion so by (I:4.2) the image, N, is $\underline{\underline{s}}$-acyclic so if it is non-zero it has some $\underline{\underline{U}}_0^i$ as image and so would $m_r F^1$. Hence F^1 would have $m_{-r}\underline{\underline{U}}_0^i = \underline{\underline{U}}_r^i$ as image but $F^1(F^1) = F^1$ and $F^1\underline{\underline{U}}_r^i = 0$ so functoriality of F^1 gives a contradiction. Thus $m_r F^1 \longrightarrow F^1(m_r M)/G^1(m_r M)$ is 0 so $m_r F^1 \longrightarrow G^1(m_r M)$. The second inclusion is proved in a similar fashion. We now put, for $s \in \mathbb{Z}$, $W^s M := m_{-s}(F^1(m_s M))$ and $W^{s-1/2} := m_{-s}(G^1(m_s M))$. As m_{-s} preserves diagonal dominoes these are admissible subobjects and we have just shown that it is a descending filtration. Property a) is clear as we have shown that $F^1(m_s M)/G^2(m_s M)$ is $\underline{\underline{s}}$-acyclic. As for b) we need to show that for any diagonal domino N, $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^k, m_1 G^1(m_{-1}N)/F^1N) = \mathrm{Hom}_\Delta(m_1 G^1(m_{-1}N)/F^1N, \underline{\underline{U}}_{-1}^k) = 0$ for all k. Put $H := m_1 G^1(m_{-1}N)/F^1N$. Then by the definition of F^1 and G^1, H contains no non-trivial $\underline{\underline{s}}$-0-torsion subobject and $m_{-1}H$ has no non-zero $\underline{\underline{s}}$-1-torsion quotient object. We then conclude by lemma 2.2.

Let us now show that a) and b) characterize $\{W^s\}$. By applying m_i it is clear that it suffices to show that W^0 and $W^{1/2}$ are characterized and we want in fact to show that they equal F^1 resp. G^1. By a) and lemma 2.2 we see that $\mathrm{Hom}_\Delta(W^s/W^{s+1/2}, \underline{\underline{U}}_0^k) = 0$ if $s \geq 1/2$. Hence $\mathrm{Hom}_\Delta(W^{1/2}, \underline{\underline{U}}_0^k) = 0$ and by lemma 2.2 $W^{1/2} \subseteq G^1$. In the same way one shows that $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^k, M/W^0) = 0$ so by lemma 2.2 M/W^0 is $\underline{\underline{s}}$-1-torsion and as $W^0/W^{1/2}$ is $\underline{\underline{s}}$-0-torsion $M/W^{1/2}$ is $\underline{\underline{s}}$-torsion and $G^1 \subseteq W^{1/2}$. The equality $W^0 = F^1$ is shown similarly.

Now $m_r(W^s(-)) = W^{s-r}(m_r(-))$ follows from the definition and the W^s are idempotent radicals as G^1 and F^1 are. By (2.2 ii) $\widetilde{D}^0(F^1(-)) = \widetilde{D}^0(-)/G^1\widetilde{D}^0(-)$ and $\widetilde{D}^0(G^1(-)) = \widetilde{D}^0(-)/F^1D^0(-)$ so iii) follows. By (I:1.11 iv) there is an s_0 such that $W^s = W^{s_0}$ if $s \geq s_0$ and W^s then has the property that $m_r W^s$ is $\underline{\underline{s}}$-0-torsion for all large r. From (2.2 iii) it then follows that $W^s(\varphi)$ is $\underline{\underline{s}}$-0-torsion for all $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ and then (I:2.2) implies that W^{s_0} is finite torsion and W^{s_0} is zero as it is without finite torsion. This gives separation and finiteness in one direction. Exhaustiveness and finiteness in

the other direction follows by duality from this and iii).

Definition 2.4. *The type of a diagonal domino* M *is the function* $\sigma : 1/2.\mathbb{Z} \longrightarrow \mathbb{N}^{\mathbb{Z}}$ *defined by*

$$\sigma(s)(i) := T^{-i,i}(W^s M/W^{s+1/2}M)$$

(cf. (0:6)). *We will say that* M *is of type* s *if* $\sigma(t)$ *equals the zero function whenever* $t \neq s$.

Thus the diagonal dominoes of type 0 are exactly the $\underline{\underline{s}}$-acyclic objects and with the aid of m_r the study of diagonal dominoes of type r is reduced to the study of those of type 0 . Similarly type $r-1/2$ is reduced to type $-1/2$.

Proposition 2.5. *A diagonal domino of type* $-1/2$ *lies in* G . *A torsion object in* G *is a diagonal domino iff*

$$\mathrm{Hom}_G(\underline{\underline{U}}_1^k, M) = 0 \quad \textit{for all} \quad k .$$

Indeed, a diagonal domino M of type $-1/2$ has no non-trivial $\underline{\underline{s}}$-1-torsion quotients so is in G and by definition $\mathrm{Hom}_G(\underline{\underline{U}}_1^k, M) = \mathrm{Hom}_{\Delta}(\underline{\underline{U}}_1^k, M) = 0$. Conversely, let $M \in G$ with $\mathrm{Hom}_G(\underline{\underline{U}}_1^k, M) = 0$ for all k . We have an exact sequence in G

$$0 \longrightarrow \tilde{H}^{-1}(M)[1] \longrightarrow M \longrightarrow \tilde{H}^0(M) \longrightarrow 0$$

so $\mathrm{Hom}_G(\underline{\underline{U}}_1^k, \tilde{H}^{-1}(M)[1]) = 0$ for all k .

Lemma 2.5.1. *Let* $N \in \Delta$ *and suppose that for some* i *and all* k $\mathrm{Hom}_{D(R)}(\underline{\underline{U}}_i^k, N[1]) = 0$. *Then* $N = 0$.

Proof : Applying $m_i(-)$ we may assume $i = 0$. By (0:4.3) we get $H^i(R_1 \bullet_R^L M)^j = 0$ if $i+j = 0$ so the spectral sequence

$$H^i(R_1 \bullet_R^L N)^j \Longrightarrow H^{i+j}(k \bullet_W^L \underline{\underline{s}}(N))$$

shows that $H^0(k \otimes^L_W \underline{\underline{s}}(N))=0$ and Nakayama's lemma gives $H^0(\underline{\underline{s}}(N)) = 0$ and, as $H^1(\underline{\underline{s}}(N))$ is torsion, $H^1(\underline{\underline{s}}(N)) = 0$ so by (I:1.4) N is $\underline{\underline{s}}$-acyclic but this together with (I:1.4 ii) and $H^i(R_1 \otimes^L_R N)^j = 0$ when $i+j = 0$ gives $R_1 \otimes^L_R N = 0$ so $N = 0$ by (0:4).

The lemma applied to $\widetilde{H}^{-1}(M)$ gives that $M \in \Delta$ and now $\mathrm{Hom}_\Delta(\underline{\underline{U}}^k_1, M) = 0$ by definition and $\mathrm{Hom}_\Delta(M, \underline{\underline{U}}^k_0) = 0$ by lemma 2.2 and the first equality gives $t^1(M) = 0$.

Corollary 2.5.1. *The category of diagonal complexes of type* $-1/2$ *is equivalent to the category of torsion coherent* F-*gauge structures* M *fulfilling the following property*

$$(*) \qquad \begin{cases} \textit{For any } i \textit{, if } x \in M^{i+1}, \ y \in M^{i-1} \textit{ and } \tau \textit{ applied to the image} \\ \textit{of } x \textit{ in } M^{\infty} \textit{ equals the image of } y \textit{ in } M^{-\infty} \textit{ then } x = y = 0 \ . \end{cases}$$

Proof : To prove the corollary it will suffice to show that for any F-gauge structure M , $\mathrm{Hom}(\underline{\underline{S}}(\underline{\underline{U}}^k_1), M) = \{(x,y) \in {}_pM^{k+1} \times {}_pM^{k-1} : \tau$ applied to the image of x in M^{∞} equals the image of y in $M^{-\infty}\}$. However, it is clear that $\underline{\underline{S}}(U^k_1)^i = k$ if $i \neq k$ and 0 if $i = k$, that $\widetilde{F} : \underline{\underline{S}}(\underline{\underline{U}}^k_1)^i \longrightarrow \underline{\underline{S}}(\underline{\underline{U}}^k_1)^{i+1}$ is the identity if $i > k$ and zero if not, that $\widetilde{V} : \underline{\underline{S}}(U^k_1)^i \longrightarrow \underline{\underline{S}}(U^k_1)^{i-1}$ is zero if $i \geq k$ and the identity if not and that τ is the p'th power map. This gives the desired description.

We will finish our study of type $-1/2$ by constructing a few examples and show that we have obtained at least all weakly simple (cf. (I, 1.5)) objects when $H = 1$.

Let $M \in \Delta$ be $\underline{\underline{s}}$-acyclic. Put $I := \{i \in \mathbb{Z} : \mathrm{Hom}_\Delta(M, \underline{\underline{U}}^i_0) \neq 0\}$ and $I' = \{i \in \mathbb{Z} : \mathrm{Hom}_\Delta(\underline{\underline{U}}^i_0, M) \neq 0\}$ and suppose that $I \cap I' = \emptyset$. Fix some J with $I \subseteq J$ and $J \cap I' = \emptyset$ and put $N := M(\varphi_J)$ where $\varphi_J(n) = 0$ if $n \notin J$ and 1 if $n \in J$. I claim that N is of type $-1/2$. Suppose that $\mathrm{Hom}_\Delta(\underline{\underline{U}}^i_1, N) \neq 0$. If $i \notin J$ then $\underline{\underline{U}}^i_1(-\varphi_J) = \underline{\underline{U}}^i_1$ and $M = N(-\varphi_J)$ so we would have a non-zero $\underline{\underline{U}}^i_1 \longrightarrow M$ but $W^{1/2}\underline{\underline{U}}^i_1 = \underline{\underline{U}}^i_1$ and $W^{1/2}M = 0$ so we get a contradiction by functoriality of $W^{1/2}(-)$. If $i \in J$ then $\underline{\underline{U}}^i_1(-\varphi_J) = \underline{\underline{U}}^i_0$ so we would have a non-

zero $\underline{\underline{U}}_0^i \longrightarrow M$ contradicting $J \cap I' = \emptyset$. Suppose now that $\mathrm{Hom}_\Delta(N, \underline{\underline{U}}_0^i) \neq 0$. If $i \notin J$ then we would get a non-zero $M \longrightarrow \underline{\underline{U}}_0^i$ contrary to $I \subseteq J$. If $i \in J$ then we would get a non-zero $M \longrightarrow \underline{\underline{U}}_{-1}^i$ but $W^{-1/2}M = M$ and $W^{-1/2}\underline{\underline{U}}_{-1}^i = 0$.

Conversely, let N be of type $-1/2$ and $J \subseteq \mathbb{Z}$ such that $M := N(-\varphi_J)$ is $\underline{\underline{s}}$-acyclic. Reversing the arguments we get $I \subseteq J$ and $I' \cap J = \emptyset$.

To $M(I,J)(\varphi_k)$ we will associate the directed graph with distinguished vertices Γ, where the graph is that of $M(I,J)$ and the vertices in $K \subseteq I$ are distinguished. Graphically we will distinguish vertices by a vertical line. So is $\overset{0\ 1\ 2\ 3\ 4}{\longrightarrow\!\!\!\times\!\!\!\longleftarrow\!\!\!\longrightarrow\!\!\!\longrightarrow}$ the graph of $M(\{0,1,2,3,4\},\{0,2,3\})(\varphi_{\{2\}})$. The condition that $M(I,J)(\varphi_k)$ be of type $-1/2$ is then that neither an undistinguished source $\ldots\leftarrow\!\bullet\!\rightarrow\ldots$, $\bullet\!\rightarrow\ldots$ or $\ldots\leftarrow\!\bullet$ nor a distinguished sink $\ldots\rightarrow\!|\!\leftarrow\ldots$, $|\!\leftarrow\ldots$ or $\ldots\rightarrow\!|$ occur.

<u>Proposition</u> 2.6. *The diagonal domino* $M = (\ldots\overset{i\ \ i+1}{\mapsto}\ldots)$ *is isomorphic to* $N = (\ldots\overset{i\ \ i+1}{\leftarrow\!|}\ldots)$.

<u>Proof</u> : Assume that we know it for $\overset{i\ \ i+1}{\mapsto}$ and $\overset{i\ \ i+1}{\leftarrow\!|}$. What we need to prove is that $(\ldots\mapsto\ldots)(-\varphi_{\{i+1\}}) \xrightarrow{\sim} (\ldots\leftarrow\!\bullet\ldots)$. Hence we may assume that i is the only distinguished vertex of $(\ldots\mapsto\ldots)$ and then we first want to prove that $M(-\varphi_{\{i+1\}})$ is $\underline{\underline{s}}$-acyclic. It is clear that there is an admissible filtration $0 \subseteq M_1 \subseteq M_2 \subseteq M$ of M s.t M_1 and M/M_2 are $\underline{\underline{s}}$-acyclic and $M_2/M_1 = \overset{i\ \ i+1}{\mapsto}$ e.g. if $M = (\ldots\overset{i\ \ \ i+1}{\rightarrow\!\!\!+\!\!\!\rightarrow\!\!\!\times\!\!\!\leftarrow}\ldots)$ then $M_1 = 0$ and $M/M_2 = (\ldots\overset{i}{\rightarrow\!\!|}) \oplus (\overset{i+1}{\leftarrow\!\bullet}\ldots)$ but $M_1 = M_1(-\varphi_{\{i+1\}})$, $M/M_2 = M/M_2(-\varphi_{\{i+1\}})$ and $M_2/M_1(-\varphi_{\{i+1\}}) = (\overset{i\ \ i+1}{\leftarrow\!\!\!-\!\!|})$ by assumption so $M(-\varphi_{\{i+1\}})$ is $\underline{\underline{s}}$-acyclic being an extension of $\underline{\underline{s}}$-acyclic objects. The following lemma then shows that $\underline{\underline{S}}(M(-\varphi_{\{i+1\}}))^j \xrightarrow{\sim} k$ if j belongs to the vertex set I of $(\ldots\overset{i\ \ i+1}{\mapsto}\ldots)$ and 0 if not.

<u>Lemma</u> 2.6.1. i) $T^{i,-i}(-)$ *is additive for short exact sequences in* $\mathbf{\Delta}$.
ii) $T^{i,-i}(-(\varphi)) = T^{i,-i}$ *for any* $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ *of finite support.*

iii) *If* $M \in \Delta$ *is* $\underline{\underline{s}}$*-acyclic then* $T^{i,-i}(M) = \dim_k \underline{\underline{S}}(M)^i$.

Indeed, let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be exact in Δ . Then by definition of Δ $H^{i-1}(M_3) \xrightarrow{\delta} H^i(M_1)$ is zero except in degree $-i+1$ where $H^i(M_1)$ is killed by a power of V so δ factors through $H^{i-1}(M_3)^{-i+1}/V^n$ for some n and so the image of δ is of finite length. Thus

$$0 \longrightarrow H^i(M_1) \longrightarrow H^i(M_2) \longrightarrow H^i(M_3) \longrightarrow 0$$

is exact modulo finite length which gives additivity for $T^{-i,i}(-)$. Now ii) is obvious as $T^{i,-i}(-)$ only depends on the R^0-complex structure and so is independent of d . Finally, iii) follows from i), dévissage and explicit calculation for $\underline{\underline{U}}_0^j$.

It is not difficult to convince oneself that $M(I,J)$ is characterized by the facts that $\dim_k M(I,J)^i = 1$ if $i \in I$, 0 if not and that for all $i \in I$ it has $M([i,i+1],\{i\} \cap J)$ as a subquotient.

Using this we see that in order to prove that $M(-\varphi_{\{i+1\}})$ is isomorphic to $N(-\varphi_{\{i+1\}})$ we can use the filtration $0 \subseteq M_1 \subseteq M_2 \subseteq M$ above and reduce down to $M = (\overset{i\ \ i+1}{\longmapsto})$ which is true by assumption.

We are hence left with $M = (\overset{i\ \ i+1}{\longmapsto})$ and by shifting we may assume $i = 0$. We know that $\overset{0\ \ 1}{\longmapsto}$ is of type $-1/2$ and if we also know that $(\longmapsto)(-\varphi_{\{1\}})$ is $\underline{\underline{s}}$-acyclic then among the three possibilities $\overset{0}{\cdot}\bullet\overset{1}{\cdot}$, $\overset{0\ \ 1}{\longmapsto}$ or $\overset{0\ \ 1}{\longleftarrow\!\bullet}$ for it we must have $\overset{0\ \ 1}{\longleftarrow\!\bullet}$ because otherwise $(\longmapsto)(-\varphi_{\{-1\}})\ (\varphi_{\{1\}})$ would not be of type $-1/2$ by the discussion above. Hence we would have $\longmapsto = \longleftarrow\!\bullet\ (\varphi_{\{1\}})$ $= \longleftarrow\!\!\dashv$. It is therefore enough to show that $Q := (\longmapsto (-\varphi_{\{1\}}))$ is $\underline{\underline{s}}$-acyclic. Now we have a short exact sequence

$$0 \longrightarrow (\overset{1}{\cdot}) \longrightarrow (\overset{0\ \ 1}{\bullet\!\longrightarrow}) \longrightarrow (\overset{0}{\cdot}) \longrightarrow 0$$

and hence an exact sequence

$$(2.6.2) \qquad 0 \longrightarrow \underline{\underline{U}}^1_{-1} \longrightarrow Q \longrightarrow \underline{\underline{U}}_1 \longrightarrow 0$$

which gives a long exact sequence

$$0 \longrightarrow H^0(\underline{\underline{s}}(Q)) \longrightarrow k \longrightarrow k \longrightarrow H^1(\underline{\underline{s}}(Q)) \longrightarrow 0 \ .$$

Assume that Q is not $\underline{\underline{s}}$-acyclic. Then both $H^0(\underline{\underline{s}}(Q))$ and $H^1(\underline{\underline{s}}(Q))$ are isomorphic to k so $0 \subsetneq W^0Q \subsetneq Q$ and $H^0(\underline{\underline{s}}(W^0(Q)) = H^0(\underline{\underline{s}}(Q))$ by (I:4.5.3). Hence the composite $\varphi : W^0Q \longrightarrow \underline{\underline{U}}_1$ induces an isomorphism after applying $\underline{\underline{s}}(-)$ so by (I:4.2) its kernel and cokernel are $\underline{\underline{s}}$-acyclic. As $\underline{\underline{U}}_1$ has no non-trivial $\underline{\underline{s}}$-acyclic quotients φ is an epimorphism and as the kernel is contained in $\underline{\underline{U}}^1_{-1}$ which has no non-trivial $\underline{\underline{s}}$-acyclic subobjects φ is a monomorphism and so an isomorphism. Therefore (2.6.2) splits and Q would be decomposable and by additivity of $(-\varphi_{\{0\}})(\varphi_{\{1\}})$ so would $\overset{0\ 1}{\bullet\!\!\longrightarrow}$ which is clearly false.

Let now $N_{m,n} := (\overset{m}{\longleftarrow\!\!\!\leftarrow\!\bullet} \ \dots \ \overset{n}{\longleftarrow\!\!\!\leftarrow\!|})$ and $N_m := (\overset{m}{\longleftarrow\!\!\!\leftarrow\!|})$. Then $N_{m,n}$ and N_m are $-1/2$-diagonal dominoes.

<u>Theorem</u> 2.7. *The category of type* $-1/2$ *diagonal dominoes is exact, weakly noetherian and weakly artinian* (cf. (I, 1.5)). *Its weakly simple objects, when* $H = 1$ *are the* $N_{m,n}$ *and* N_m .

The first part is clear. Let $H = 1$ and let M be a weakly simple type $-1/2$ diagonal domino. By artinian induction using (I:1.11) we may assume that every type $-1/2$ subobject such that the quotient of M by this subobject is non-zero and without finite torsion contains some $N_{m,n}$ or N_m as admissible subobject. I claim first that M is killed by p . Indeed, the kernel and image of multiplication by p <u>in</u> G are of type $-1/2$ by Proposition 2.5 being subobjects of M . Hence the kernel of p is an admissible subobject of M . It is not 0 however ; as $\underline{\underline{s}}(M)$ is torsion $\underline{\underline{S}}(M)$ is torsion and p is not injective on M . As M is weakly simple

the kernel of p is all of M. Now $\mathrm{Hom}_\Delta(M,\underline{\underline{U}}_1^i) \neq 0$ for some i because if not $0 = W^1M = M$ by (2.2). Let $\varphi : M \longrightarrow \underline{\underline{U}}_1^i$ be non-zero. By (0:4.2) and (I:1.5.1) every non-zero proper subobject of $\underline{\underline{U}}_1^i$ is isomorphic to $\underline{\underline{U}}_j^i$ for $j \leq 0$ but $\mathrm{Hom}_\Delta(M,\underline{\underline{U}}_j^i) = 0$ if $j \leq 0$ by (2.2) so $\varphi : M \longrightarrow \underline{\underline{U}}_1^i$ is an epimorphism. Let N be the kernel of φ. Suppose first that $W^{1/2}N = 0$. Then N is $\underline{\underline{s}}$-1-torsion by definition. We get a short exact sequence by (I:1.4)

$$0 \longrightarrow H^0(\underline{\underline{s}}(M)) \longrightarrow k \longrightarrow H^1(\underline{\underline{s}}(N)) \longrightarrow 0 .$$

As M is not $\underline{\underline{s}}$-1-torsion $H^0(\underline{\underline{s}}(M)) \neq 0$ so N is $\underline{\underline{s}}$-acyclic. Now $M \in G$ so $H^1(\underline{\underline{S}}(M)^i) = 0$, $N[1] \in G$ so $H^0(\underline{\underline{S}}(N)^i) = 0$ and as $H^0(\underline{\underline{S}}(\underline{\underline{U}}_1^i)^i) = 0$ the exact sequence $H^0(\underline{\underline{S}}(N)^i) \longrightarrow H^0(\underline{\underline{S}}(M)^i) \longrightarrow H^0(\underline{\underline{S}}(\underline{\underline{U}}_1^i)^i)$ gives $H^0(\underline{\underline{S}}(M)^i) = 0$. Thus, in the notation from above, $M(-\varphi_{\{i\}})$ is $\underline{\underline{s}}$-acyclic and indecomposable and killed by p as M is. As M is of type $-1/2$ we see by the discussion above that the only source of the graph associated to $M(-\varphi_{\{i\}})$ is at i and so $M = (\dots \leftarrow\leftarrow\bullet\rightarrow\rightarrow \dots)$. By using (2.6) repeatedly we can move the sink and the distinguished vertex to the right and so for some m,n $M = (\overset{m}{\bullet}\leftarrow\leftarrow\bullet \dots \leftarrow\leftarrow\overset{n}{\bullet}) = N_{m,n}$. We may therefore assume that $W^{1/2}N \neq 0$ and as $M/W^{1/2}N$ is an extension of $N/W^{1/2}N$ by $\underline{\underline{U}}_1^i$ it is without finite torsion. Further, $W^{1/2}N$ is of type $-1/2$ and so there is some admissible $\bar{N} \subseteq W^{1/2}N$ with $\bar{N} \xrightarrow{\sim} N_{m,n}$ or N_m. Put $N' := N/\bar{N}$ and $M' := M/\bar{N}$. Now $W^{1/2}M' = M'$ as M' is a quotient of M. However, M' is not of type $-1/2$ as M is weakly simple and so, for some k, there is a non-zero $\underline{\underline{U}}_1^k \longrightarrow M'$. As $W^{1/2}N' = W^{1/2}N/\bar{N}$ is of type $-1/2$ $\mathrm{Hom}_\Delta(\underline{\underline{U}}_1^k,N') = 0$ and hence the composite $\underline{\underline{U}}_1^k \longrightarrow M' \longrightarrow \underline{\underline{U}}_1^i$ is non-zero. By (0:4.3), and [Il-Ra:I.3.7] and modification by m_1, $\mathrm{Hom}_\Delta(\underline{\underline{U}}_1^k,\underline{\underline{U}}_1^i) = 0$ if $k \neq i$ whence $k = i$ and by (0:4.2) the composite is an isomorphism whence $M' = N' \oplus \underline{\underline{U}}_1^i$. Let N'' be the inverse image in M of the $\underline{\underline{U}}_1^i$-factor. Thus N'' is an extension of $\bar{N}$ by $\underline{\underline{U}}_1^i$ whence $W^{1/2}N'' = N''$ and being a subobject of M, $W^1N'' = 0$ so N'' is of type $-1/2$. On the other hand, N' is a quotient of M so $W^{1/2}N' = N'$, so $N' = W^{1/2}N/\bar{N}$ and so of type $-1/2$ by assumption. As $N' = M/N''$ and $N'' \neq 0$ $M = N''$ as it

is weakly simple. We have therefore obtained a short exact sequence

(2.7.1) $$0 \longrightarrow \bar{N} \longrightarrow M \longrightarrow \underline{\underline{U}}_1^i \longrightarrow 0 .$$

We now have two cases

i) $\bar{N} = M_{m,n}$. Suppose $i \in [m,n]$. By repeated use of (2.6) we get $\bar{N} = (\overset{m}{\twoheadleftarrow}\bullet \cdots \overset{i}{\leftrightarrow} \cdots \overset{n}{\rightarrow})$ and so (2.7.1) shows that $M(-\varphi_{\{i\}})$ is an extension of something $\underline{\underline{s}}$-acyclic by something $\underline{\underline{s}}$-acyclic and so is $\underline{\underline{s}}$-acyclic and is indecomposable and killed by p as M is. However, $T^{i,-i}(M(-\varphi_{\{i\}})) = T^{i,-i}(M) = 2$ which contradicts (1.3). Hence $i \notin [m,n]$ and so $M(-\varphi_{\{i,n\}})$ is an extension of $\underline{\underline{s}}$-acyclic by $\underline{\underline{s}}$-acyclic and so $\underline{\underline{s}}$-acyclic and indecomposable and killed by p . Clearly then $i = m-1$ or $n-1$ and the graph of $M(-\varphi_{\{i,n\}})$ has vertex set $[m,n] \cup \{i\}$ and sources at most i and n and no sink at i or n as M is of type $-1/2$. This gives 3 possibilities for M : $\overset{m-1}{\longmapsto} \cdots \overset{j}{\twoheadleftarrow}\bullet\cdots \overset{n}{\leftarrow\!\shortmid}$, $\overset{m}{\twoheadleftarrow}\bullet \cdots \overset{n+1}{\leftarrow\!\shortmid}$ or $\overset{m}{\twoheadleftarrow}\bullet \cdots \overset{n+1}{\twoheadleftarrow\!\shortmid}$.
In the first we use (2.6) repeatedly to get either $\overset{m-1}{\twoheadleftarrow}\bullet \cdots \overset{j}{\twoheadleftarrow}\bullet \cdots \overset{n}{\leftarrow\!\shortmid}$ or $\overset{m-1}{\twoheadleftarrow\!\shortmid}$. The first is an extension of $\overset{m-1}{\twoheadleftarrow}\bullet \cdots \overset{j}{\leftarrow\!\shortmid}$ by $\overset{j+1}{\twoheadleftarrow}\bullet \cdots \overset{n}{\leftarrow\!\shortmid}$ and so not weakly simple. The second is N_{m-1} . The second possibility is $N_{m,n+1}$ and the third is either N_m or by (2.6) $\overset{m}{\twoheadleftarrow}\bullet \cdots \overset{n+1}{\twoheadleftarrow\!\shortmid}$ which is an extension of $\overset{m}{\twoheadleftarrow}\bullet \cdots \overset{n-1}{\leftarrow\!\shortmid}$ by $\overset{m+1}{\leftarrow\!\shortmid}$ and so is not weakly simple.

ii) $\bar{N} = N_m$. As in i) one shows that $i = m-1$ or $m+3$ and we get the following possibilities $\overset{m}{\longmapsto\!\twoheadleftarrow\!\shortmid}$; $\overset{m}{\twoheadleftarrow\!\twoheadleftarrow\!\shortmid}$. The first is isomorphic to the second shifted one step by (2.6) and the second is an extension of $\overset{m}{\leftarrow\!\shortmid}$ by $\overset{m+2}{\leftarrow\!\shortmid}$ and so not weakly simple.

It remains to show that the $N_{m,n}$ and N_m are weakly simple. Now $H^0(\underline{\underline{s}}(-))$ is exact on diagonal dominoes of type $-1/2$ and zero only on the zero object. As $\dim H^0(\underline{\underline{s}}(N_{m,n})) = 1$ this shows that $N_{m,n}$ is weakly simple. On the other hand is N_m were not weakly simple then some weakly simple admissible subquotient M' would have $\sum_i T^{i,-i}(M') = 1$ as $\sum_i T^{i,-i}(N_m) = 3$ and it would be among the $N_{m,n}$ and N_m but $\sum T^{i,-i}(-) \geq 2$ on those.

Dominoes

3. We will now see how dominoes fit in.

Proposition 3.1. *The dominoes are exactly the diagonal dominoes of level* $[0,1]$.

Proof : From (I:2.2.2) it is clear that a domino is a diagonal domino and it is of level $[0,1]$. Conversely, again by (I:2.2.2), a diagonal domino M of level $[0,1]$ has $HW(H^0(M)) = 0$ so $H^0(M)$ is by definition a domino and because M is of level $[0,1]$ $M = H^0(M)$.

From (2.8) or (2.5.1) it follows that there are no dominoes of half integer type. It is also clear from the description as F-gauge structures taht a domino of type 0 is a direct sum of $\underline{\underline{U}}_0$:s (cf. III:1.2). Hence

Proposition 3.2. *Any domino* M *has a unique filtration*

$$0 \subseteq \ldots \subseteq W^iM \subseteq W^{i-1}M \subseteq \ldots \subseteq M$$

s.t. $W^iM/W^{i+1}M$ *is isomorphic to a sum of* $\underline{\underline{U}}_i$:s .

The type of M can then be identified with a function $\sigma : \mathbb{Z} \longrightarrow \mathbb{N}$ s.t. $\sigma(i) = \dim W^iM/W^{i+1}M$.

Proposition 3.3. *Let* M *be a domino of type* σ

$$\operatorname{lgth} H^0(M,F^nd) = \sum_{i\in\mathbb{N}} i\ \sigma(i-n)$$

$$\operatorname{lgth} H^1(M,F^nd) = \sum_{i\in\mathbb{N}} i\ \sigma(-i-n) \qquad \textit{where} \quad F^{-n}d = dV^n \quad n > 0 .$$

Indeed, by shifting à la Nygaard we may assume $n = 0$ and as $H^0(\underline{\underline{U}}_i,d) = 0$ if $i \leq 0$ and $H^1(\underline{\underline{U}}_i,d) = 0$ if $i \geq 0$ we get $H^0(M,d) = H^0(W^0M,d)$ and $H^1(M,d) = H^1(M/W^1M,d)$ and on dominoes with $W^0(-) = \mathrm{id}$ $H^0(-,d)$ is exact and on dominoes with $W^0(-) = 0$ $H^1(-,d)$ is exact so we reduce to

$M = \underline{\underline{U}}_i$ where we compute directly.

Mazur-Ogus objects

4. Recall, (I, 4.3), that a diagonal complex M is said to be Mazur-Ogus iff $\mathrm{Hom}_\Delta(M,N) = \mathrm{Hom}_\Delta(N,M) = 0$ for all $\underline{\underline{s}}$-acyclic N.

Proposition 4.1. *The following are equivalent for* $M \in \Delta$

i) M *is Mazur-Ogus.*

ii) $H^i(R_1 \otimes^L_R M)^j \Longrightarrow H^{i+j}(k \otimes^L_W \underline{\underline{s}}(M))$ *degenerates,* $H^1(\underline{\underline{s}}(M)) = 0$ *and* $H^0(\underline{\underline{s}}(M))$ *is torsion free.*

iii) $b_0(M) = \sum_i h^{i,-i}(M)$ (*cf.* (0:6.2)).

Indeed, i) $\Longrightarrow$ ii) is (I:4.4). Now $\sum_i h^{i,-i}(M) \leq b_0^{DR}(M)$ with equality iff $H^i(R_1 \otimes^L_R M)^j = E_1^{j,i} = E_\infty^{j,i}$ and $b_0(M) \leq b_0^{DR}(M)$ with equality iff $H^0(\underline{\underline{s}}(M))$ and $H^1(\underline{\underline{s}}(M))$ are without torsion. This and (I:1.4) gives ii) $\Longleftrightarrow$ iii). Finally, ii), (0:4.3) gives $\mathrm{Hom}_\Delta(\underline{\underline{U}}_0^i, M) = \mathrm{Hom}_\Delta(M, \underline{\underline{U}}_0^i) = 0$ for all i when $H = 1$ and (II:1.2.1) gives i). The modifications needed for $H \neq 1$ are left to the reader.

Theorem 4.2. i) $M \in G$ *is torsion free in* G *iff* $M \in \Delta$ *and in the notations of* (I:4.5) $G^2 = M$ *and* G^1 *is* $\underline{\underline{s}}$-*acyclic.*

ii) *If* N *is a virtual* F-*crystal of finite type then* $L\hat{t}(\mathrm{Hodge}(N))$ *is a Mazur-Ogus object in* Δ.

iii) *If* M *in* Δ *is Mazur-Ogus then* $M \in G$, $\underline{\underline{S}}(M)$ *is torsion free and* $\underline{\underline{S}}(M) = \mathrm{Hodge}(\underline{\underline{S}}(M)^{-\infty} \otimes K)$ *where* $\underline{\underline{S}}(M)^{-\infty} \otimes K$ *is considered as a virtual* F-*crystal as in* II.

Corollary 4.2.1. *If* $M \in \Delta$ *is Mazur-Ogus then the Hodge filtration on* $\underline{\underline{s}}(M)/p$ *is given by the images of the* $\underline{\underline{S}}(M)^i$.

We may assume $H=1$. The corollary follows immediately from (II:2.4.1). Let us now show that if $M \in \Delta_{MO}$ then $M \in G$ and is torsion free. By lemma 2.2 $G^2M = M$ so by definition $M \in G$. Let N be the torsion subobject of M in G. As $M \in \Delta$ the composite $\tilde{\tau}_{<-1}N \longrightarrow N \longrightarrow M$ is zero so $N \in \Delta$. By (4.1) M is without $\underline{\underline{s}}$-torsion so $\overline{N}$ is $\underline{\underline{s}}$-1-torsion and hence zero as it is in G. Suppose now that $M \in \Delta$ with $G^2M = M$ and G^1M is $\underline{\underline{s}}$-acyclic. Then we get a distinguished triangle

$$\longrightarrow G^1 \longrightarrow M \longrightarrow M/G^1 \longrightarrow G^1[1]$$

and M/G^1 is Mazur-Ogus. Taking G-cohomology gives us an exact sequence $(G^2M = M \Longrightarrow M \in G)$ in G

$$0 \longrightarrow M \longrightarrow M/G^1 \longrightarrow G^1[1] \longrightarrow 0$$

so M is a subobject of M/G^1 which we have just shown to be torsion free. Conversely let $M \in G$ be torsion free. As $\tau_{\leq -1}M$ is always torsion $M \in \Delta$. Now $W^{1/2}(F^1M)$ is a torsion subobject of M in G so zero. Therefore F^1M is $\underline{\underline{s}}$-acyclic and as $M \in G$ implies $G^2M = M$ we get $G^1M = F^1M$. We have thus proved i).

Let N be a virtual F-crystal of finite type. Then $\mathrm{Hodge}(N)$ is a torsion free coherent F-gauge structure and so $M := L\hat{t}(\mathrm{Hodge}(N))$ is torsion free in G. As G^1M is $\underline{\underline{s}}$-acyclic, $N = \underline{\underline{S}}(M)^{-\infty} = \underline{\underline{S}}(M/G^1)^{-\infty}$. Hence adjunction gives us a morphism $\underline{\underline{S}}(M/G^1) \longrightarrow \mathrm{Hodge}(N)$ such that the composite $\mathrm{Hodge}(N) = \underline{\underline{S}}(M) \longrightarrow \underline{\underline{S}}(M/G^1) \longrightarrow \mathrm{Hodge}(N)$ is the identity. Hence $0 \longrightarrow G^1 \longrightarrow M \longrightarrow M/G^1 \longrightarrow 0$ splits but as $M \in G$ and G^1 is $\underline{\underline{s}}$-acyclic, $G^1 = 0$ and $L\hat{t}(\mathrm{Hodge}(N))$ is Mazur-Ogus. This proves ii).

Let finally M be Mazur-Ogus. We have seen that $\underline{\underline{S}}(M)$ is torsion free and adjunction gives us a morphism $\underline{\underline{S}}(M) \longrightarrow \mathrm{Hodge}(N)$ where $N := \underline{\underline{S}}(M)^{-\infty}$. Applying $L\hat{t}$ gives us $\varphi : M \longrightarrow L\hat{t}(\mathrm{Hodge}(N))$ inducing an isomorphism on applying $\underline{\underline{s}}(-)$ so the kernel and cokernel of φ is $\underline{\underline{s}}$-acycling. However M is Mazur-Ogus so the kernel is zero and we have just seen that $L\hat{t}(\mathrm{Hodge}(N))$

is Mazur-Ogus so the cokernel is zero. Applying $\underline{\underline{S}}$ we get $\underline{\underline{S}}(M) = \mathrm{Hodge}(\underline{\underline{S}}(M^{-\infty}))$.

The next corollary is so important that we give it as a theorem.

Theorem 4.3. *The functors* $\underline{\underline{S}}(-)^{-\infty}$ *and* $t'(\mathrm{Hodge}(-))$ *gives inverse equivalences between the category of Mazur-Ogus diagonal complexes and the category of virtual* F*-crystals of finite type.*

Proof : Combine (4.2) and (II:4.6).

We will now use the explicit form $t'(\mathrm{Hodge}(-))$ to obtain a result on the spectral sequence $H^i(M)^j \Longrightarrow H^{i+j}(\underline{\underline{s}}(M))$. Before we can do this, however, we will need the following lemma.

Lemma 4.4. *Let* M *be a coherent* F*-gauge structure of level* I . *Then* $t'(M)^i$ *is complete (and so separated) in the standard topology for all* i .

Proof : Recall that the standard topology on an R-module N is the topology defined by $\{dV^nN+V^nN\}$. We can find a resolution $F^1 \xrightarrow{\Phi} F^0 \longrightarrow M \longrightarrow 0$ where F^0 and F^1 are finite sums of G_r:s . As M is coherent and F^i p-torsion free we get an exact sequence $\hat{F}^1 \xrightarrow{\Phi} \hat{F}^0 \longrightarrow M \longrightarrow 0$ where $(\hat{-})$ denotes p-adic completion. This gives an exact sequence $t'(\hat{F}^1) \xrightarrow{\Phi} t'(\hat{F}^0) \longrightarrow t'(M) \longrightarrow 0$ and as $t'(\hat{F}^j)^i$ are complete in the standard topology by the proof of (II:4.6) it suffices to show that the image of Φ^i is closed. In module degree $-i$ this is clear as $t'(\hat{F}^j)^i$ in degree $-i$ are of finite type over $\hat{R}^0$ resp. $W\{F,F^{-1}\}$ which are noetherian. In degree $-i+1$ we have $t'(\Phi)^i : (\hat{R}^1)^k \longrightarrow (\hat{R}^1)^\ell$ and furthermore as Φ is the completion of a morphism between G_r:s $t'(\Phi)^i$ is a matrix with elements in R^0 . Let V^j be the highest power of V which occurs in this matrix. Then $t'(\Phi)^i$ takes $M^+ = (\sum_{r\geq 0} WF^rd)^k$ to $\sum_{r\geq 0} WF^{r-j}d = N^{\geq -j}$ where $F^{-s}d = dV^s$, $s \geq 0$ and the $\sum$-sign means series convergent in the p-adic topology. Note that $\hat{R}^1 = \prod_{r>0} WdV^r \oplus \sum_{r\geq 0} WF^rd$ with the $\sum$-sign having the

same meaning as before and the product resp. p-adic series-topology on the first resp. second factor. Now $t'(\Phi)^i$ restricted to M^+ has image in $N^{\geq -j}$ and M^+ resp. $N^{\geq -j}$ are of finite type over $W\{F\}$ which is noetherian. Hence the image of M^+ is closed in $N^{\geq -j}$ and so in $(\hat{R}^1)^\ell$ as $N^{\geq -j}$ is. Therefore the image of $t'(\Phi)^i$ is closed iff the image of $(\hat{R}^1)^k/M^+ \longrightarrow (\hat{R}^1)^\ell/\mathrm{Im}(M^+)$ is closed. Now $(\hat{R}^1)^k/M^+$ is profinite so this image is closed.

Let now $M = (V,\underline{F},N)$ be a virtual F-crystal. We will say that M is divisible by p^i, $i \in \mathbb{Z}$, if $p^{-i}\underline{F}$ takes N into itself. It is clear that for every i there is a unique largest sub-virtual F-crystal of M which is divisible by p^i which we will denote d_iM. Clearly the d_iM form a descending filtration. Let now $N \in G$ and consider $V = \underline{\underline{S}}(N)^{-\infty} \otimes_W K$ with its associated structure of virtual F-crystal. We have a spectral sequence

$$H^i(N)^j \Longrightarrow H^0(\underline{\underline{s}}(N)) = \underline{\underline{S}}^{-\infty}(N) .$$

Let F_iN be the image in V of the $\mathrm{Fil}^i H^0(\underline{\underline{s}}(N))$ coming from this spectral sequence. From the definitions it is clear that $(V,\underline{F},F_iN)$ is divisible by p^i so $F_iN \subseteq d_iM$ where $M = (V,\underline{F}\ \underline{\underline{S}}(N)^{-\infty}/\mathrm{tors})$.

<u>Theorem</u> 4.5. *Let* $N \in G$ *be Mazur–Ogus. Then* $F_iN = d_iN$ *for all* i .

Indeed, one direction has already been noted, so we need only show that the composite $d_iM \longrightarrow N/F_iN$ is zero. If we put $T := \underline{\underline{S}}(N)$, then we may assume that $N = t'(T)$ and we want to show that if $m \in d_iM$ then $\beta_j \otimes m$ is 0 in $t'(T)$ for all $j < i$. As $d.M$ is a decreasing filtration it will suffice to do it for $j = i-1$. It is clear that $d_iM \subseteq M^i \subseteq M^{i-1}$ so that $m = \tau n$ where $n \in M^{i-1}$ and $\tau : M^i \longrightarrow M^{-\infty}$. Clearly also $n \in \bigcap_\ell \underline{F}^{-\ell}p^{\ell i}M^{i-1}$. As $\beta_i \otimes m = p^i(\beta_i \otimes n)$ it suffices to show that $\beta_i \otimes n = 0$. Suppose that N is of amplitude $[r,s]$ with $t := s-r$. We have $\underline{F}n = p^i n'$ for some $n' \in \bigcap_\ell \underline{F}^{-\ell}p^{\ell i}M^{i-1}$ so $\tilde{F}^i\,\tilde{F}^{t-i}n = \tilde{F}^i\tilde{V}^i n'$ and as $\tilde{F}^i$ is injective we get

$\tilde{F}^{t-i}n = \tilde{V}^i n'$. (Here we let $\tilde{F}$ resp. $\tilde{V}$ also denote the composite $\tilde{F}: M^{s-1} \xrightarrow{\tilde{F}} M^s \longrightarrow M^{\infty} \longrightarrow M^{-\infty} \longrightarrow M^r$ and similarly for $\tilde{V}$.). Using the relations in $t'(T)$ we get $\beta_i \circledast n = V(\beta_i \circledast n')$. Iterating we get $\beta_i \circledast n \in \bigcap_{\ell} V^{\ell} t'(T)^{i,-i}$ which is zero by lemma 4.4.

Theorem 4.6. *The category of diagonal Mazur-Ogus complexes is equivalent to the category of tuples* $(M_1, M_2, N_1, N_2, \psi, \varphi, \rho)$ *where* M_1 *and* M_2 *are torsion free diagonal Hodge-Witt complexes,* N_1 *and* N_2 *are diagonal dominoes with* $W^0 N_1 = 0$ *and* $W^{1/2} N_2 = N_2$, ψ *is a monomorphism* $M_1 \longrightarrow M_2$ *, with finite torsion cokernel,* φ *is a monomorphism* $N_1 \longrightarrow N_2$ *with finite torsion cokernel and* ρ *is an isomorphism* Coker $\psi \longrightarrow$ Coker φ .

Proof : Let $M \in \Delta_{M0}$ and put $M_1 := HW(M)$ (cf. (I:2)), $M_2 := D(HW(D(M)))$, so M_2 is the largest torsion free Hodge-Witt quotient of M , $N_1 := \mathrm{Ker}(M \longrightarrow M_2)$ and $N_2 := M/M_1$. Let ψ be the composite $M_1 \hookrightarrow M \longrightarrow M_2$ and φ the composite $N_1 \hookrightarrow M \longrightarrow N_2$. Then Coker $\psi = M/N_1/\mathrm{Im}\, M_1 = M/N_1 + M_1 = M/M_1 . \mathrm{Im}\, N_1 =$ Coker φ . Let ρ be this identification.

Let us now verify that $(M_1, M_2, N_1, N_2, \psi, \varphi, \rho)$ fulfills the conditions of the theorem. It is clear that N_1 and N_2 are torsion objects without finite torsion subobjects and hence diagonal dominoes. Now $N_1 \cap M_1$ is a diagonal domino as a subobject of N_1 and Hodge-Witt as a subobject of M_1 and so zero. However, $N_1 \cap M_1 = \mathrm{Ker}\,\psi = \mathrm{Ker}\,\varphi$ and so ψ and φ are monomorphisms. Coker φ = Coker ψ is torsion being a quotient of N_2 and Hodge-Witt being a quotient of M_2 and so finite torsion. Finally, $W^0 N_1 \subseteq W^0 M = 0$ and $0 = M/W^{1/2}M \twoheadrightarrow N_2/W^{1/2}N_2$. Conversely, let $(M_1, M_2, N_1, N_2, \psi, \varphi, \rho)$ have the properties of the theorem. Let N' be a cokernel of the morphism

$$\mathrm{Coker}\,\psi \xrightarrow{\rho} \mathrm{Coker}\,\varphi$$

and M the pullback of the diagram

$$\begin{array}{ccc} & & N_2 \\ & & \downarrow \\ M_2 & \longrightarrow & N' \end{array}$$

with morphisms the composites $N_2 \twoheadrightarrow \mathrm{Coker}\,\varphi \xrightarrow{\sim} N'$ resp. $M_2 \twoheadrightarrow \mathrm{Coker}\,\psi \xrightarrow{\sim} N'$. I claim that M is Mazur-Ogus. Indeed, for this it will suffice to show that $\mathrm{Hom}_\Delta(A,M) = 0$ and $\mathrm{Hom}_\Delta(M,A) = 0$ for any $\underline{\underline{s}}$-acyclic A. By assumption we get a short exact sequence $0 \to M_1 \to M \to N_2 \to 0$ and $\mathrm{Hom}_\Delta(M_1,A) = \mathrm{Hom}_\Delta(N_2,A) = 0$ the first as $HW(M_1) = M_1$ and $HW(A) = 0$ and the second as $W^{1/2}N_2 = N_2$ and $W^{1/2}A = 0$. Again by assumption we get an exact sequence $0 \to N_1 \to M \to M_2 \to 0$ and $\mathrm{Hom}_\Delta(A,N_1) = \mathrm{Hom}_\Delta(A,M_2) = 0$ the first as $W^0A = A$ and $W^0N_1 = 0$, the second because M_2 is Mazur-Ogus.

It is clear that the two constructions just described give functors which are inverses to each other.

<u>Corollary</u> 4.6.1. *Let* $M \in \Delta_{MO}$. *Then* $D(M) = \widetilde{D}^0(M) \in \Delta_{MO}$ *and if* $M = (M_1,M_2,N_1,N_2,\psi,\varphi,\rho)$ *then* $D^0(M) = (\widetilde{D}^0(M_2),\widetilde{D}^0(M_1),\widetilde{D}^0(N_2),\widetilde{D}^0(N_1),\widetilde{D}^0(\psi),\widetilde{D}^0(\varphi),\widetilde{D}^1(\rho)^{-1})$. *In particular, if* $M \xrightarrow{\sim} D^0(M)$ *is a perfect pairing then* $v_p(\mathrm{disc}\ \underline{\underline{s}}(M_1)) = 2.\mathrm{lgth}\ H^0(\underline{\underline{s}}(N_2))$ *where* $v_p(-)$ *is the* p*-adic valuation on* W *with* $v_p(p) = 1$.

<u>Proof</u> : As M contains non non-trivial finite torsion subobject $D(M) = \widetilde{D}^0(M)$ and if there is a non-zero $N \to D(M)$ (resp. $D(M) \to N$) with N $\underline{\underline{s}}$-acyclic then, by duality, there is a non-zero $M \to D(N)$ (resp. $D(N) \to M$) but $D(N)$ is $\underline{\underline{s}}$-acyclic in Δ and M is Mazur-Ogus so we get a contradiction. That $\widetilde{D}^0(M_2) \hookrightarrow \widetilde{D}^0(M)$ is the maximal Hodge-Witt subobject is clear by duality and similarly $\widetilde{D}^0(M) \to \widetilde{D}^0(M_1)$ is the maximal Hodge-Witt-quotient-object. This clearly gives the second part. As for the final part we have $v_p(\mathrm{disc}(\underline{\underline{s}}(M_1))) = \mathrm{lgth}\ \underline{\underline{s}}(\widetilde{D}^0(M_1)/M_1)$ as $\underline{\underline{s}}(\widetilde{D}^0(M_1)) = \mathrm{Hom}(\underline{\underline{s}}(M_1),W)$. Now $\widetilde{D}^0(M_1) = M_2$ and $M_2/M_1 = N_2/N_1 = N_2/\widetilde{D}^0(N_2)$ and so

$\mathrm{lgth}\ \underline{\underline{s}}(\widetilde{D}^0(M_1)/M_1) = \mathrm{lgth}\ \underline{\underline{s}}(N_2) - \mathrm{lgth}\ \underline{\underline{s}}(\widetilde{D}^0(N_2)) = \mathrm{lgth}\ H^0(\underline{\underline{s}}(N_2)) + \mathrm{lgth}$ $H^1(\underline{\underline{s}}(\widetilde{D}^0(N_2))) = 2.\mathrm{lgth}\ H^0(\underline{\underline{s}}(N_2))$, the last by duality.

Inspired by (4.6) we introduce the following notation

Definition 4.7. *Let* $M \in \Delta_{M0}$. *Put* p-M *the* p-*torsion subobject of* M , M-HW := M/p-M *and* M-p := M/HW(M).

It is clear that the (M_1,M_2,N_1,N_2) are the (HW(M),M-HW,p-M,M-p).

Let again $M \in \Delta_{M0}$ and consider the spectral sequences

$$(4.8)_n \qquad E_1^{j,i} = H^i(R_n \otimes^L_R M) \Longrightarrow H^{i+j}(W/p^n \otimes^L_W \underline{\underline{s}}(M)) \ .$$

As $W/p^n \otimes^L_W \underline{\underline{s}}(M)$ is concentrated in degree 0 and $E_1^{j,i}$ is concentrated in total degrees -1, 0 and 1 , by (I:1.4), it is clear that $(4.8)_n$ degenerates iff $H^i(R_n \otimes^L_R M)^j = 0$ for $i+j = -1$ or 1 .

Lemma 4.9. i) *Let* $M \in \Delta$. *Then* $H^{i+1}(R_n \otimes^L_R M)^{-i} = 0$ *for all* i *iff* $\mathrm{Hom}_\Delta(M,\underline{\underline{U}}^j_{n-1}) = 0$ *for all* j .

ii) *Let* $M \in \Delta_{M0}$. *Then* $(4.8)_n$ *degenerates iff* $\mathrm{Hom}_\Delta(\underline{\underline{U}}^j_{-n+1},M) = \mathrm{Hom}_\Delta(M,\underline{\underline{U}}^j_{n-1})=0$ *for all* j .

Indeed, $\mathrm{Hom}_W(R_n \otimes^L_R M,K/W) = \mathrm{RHom}_R(M,\mathrm{Hom}_W(R_n,K/W)) = \mathrm{RHom}_R(M,\check{R}_n)(1)$, $\check{R}_n := \mathrm{Hom}_W(R_n,K/W)(-1)$, so that $H^{i+1}(R_n \otimes^L_R M)^{-i} = 0$ for all i iff there are no non-zero morphisms $M \longrightarrow \check{R}_n(j)[-j]$ for all j . By shifting we are then reduced to showing that for any $N \in \Delta$ $\mathrm{Hom}_\Delta(N,\check{R}_n) = 0$ iff $\mathrm{Hom}_\Delta(N,\underline{\underline{U}}_{n-1}) = 0$. Suppose we know that $\check{R}_n$ is a domino whose type σ has $\sigma(i) = 0$ if $i > n-1$ and $\sigma(n-1) = 1$. Then $\underline{\underline{U}}_{n-1} \hookrightarrow \check{R}_n$ so one implication is clear. If on the other hand $\mathrm{Hom}_\Delta(N,\underline{\underline{U}}_{n-1}) = 0$ then as $\underline{\underline{U}}_i \hookrightarrow \underline{\underline{U}}_{n-1}$ for all $i \leq n=1$ $\mathrm{Hom}_\Delta(N,\underline{\underline{U}}_i) = 0$ so by dévissage $\mathrm{Hom}_\Delta(N,T) = 0$ for any domino T with $\sigma(i) = 0$ when $i > n-1$.

That $\check{R}_n$ is a domino is easily checked and by (III:3.2) it only remains to show that $dV^{n-1} : \check{R}^0_n \longrightarrow \check{R}^1_n$ is injective and that $\dim_k \operatorname{Ker} dV^{n-2} = 1$ or by duality that $dV^{n-1} : R^0_n \longrightarrow R^1_n$ is surjective and $\dim_k \operatorname{Coker} dV^{n-2} = 1$ which is easily verified by inspection. We have thus proved i). Now ii) follows from i), the discussion preceding the lemma and duality.

Proposition 4.10. *Let* $M \in \Delta_{M0}$. *Then* $(4.8)_n$ *degenerates iff* $W^{1-n}(p\text{-}M) = 0$ *and* $W^{n-1/2}(M\text{-}p) = M\text{-}p$.

Indeed, $HW(M)$ and $M\text{-}HW$ are torsion free Hodge-Witt so for them $(4.8)_n$ degenerates for any n . Hence by lemma 4.9 $\operatorname{Hom}_\Delta(\underline{\underline{U}}^j_{-n+1}, M) = \operatorname{Hom}_\Delta(\underline{\underline{U}}^j_{-n+1}, p\text{-}M)$ and $\operatorname{Hom}_\Delta(M, \underline{\underline{U}}^j_{n-1}) = \operatorname{Hom}_\Delta(M\text{-}p, \underline{\underline{U}}^j_{n-1})$ for all j and it follows easily from the characterization of the type filtration that for a diagonal domino N $W^{1-n}(N) = 0$ $(W^{n-1/2}(N) = N)$ iff $\operatorname{Hom}_\Delta(\underline{\underline{U}}^j_{-n+1}, N) = 0$ (resp. $\operatorname{Hom}_\Delta(N, \underline{\underline{U}}^j_{n-1}) = 0$) for all j . We then conclude by (4.9 ii).

Corollary 4.10.1. *Let* $M \in \Delta_{M0}$.

i) *If* $(4.8)_n$ *degenerates then* $(4.8)_m$ *degenerates if* $m \leq n$.

ii) *If* $(4.8)_n$ *degenerates and* M *is not Hodge-Witt then* $2n-1 \leq \max_i m^{i,-i}(M)$ (*cf.* (0:6.1)).

Indeed, i) follows from the fact that $W^\cdot$ is a decreasing filtration. It is clear from (2.2 i) that if $N \neq 0$ is a diagonal domino with $W^{n-1/2}N = N$ then there is a monomorphism $\underline{\underline{U}}^j_{n-1} \hookrightarrow N$ for some j . Now by assumption $M\text{-}p \neq 0$ and so by shifting we may assume that there is a monomorphism $\underline{\underline{U}}_{n-1} \hookrightarrow M\text{-}p$ and as $\dim_k \operatorname{Hom}_\Delta(\underline{\underline{U}}_{-n+1}, \underline{\underline{U}}_{n-1}) = 2n-1$ we see that $\dim_k \operatorname{Hom}_\Delta(\underline{\underline{U}}_{-n+1}, M\text{-}p) \geq 2n-1$. By (4.9) $\operatorname{Hom}_\Delta(\underline{\underline{U}}_{-n+1}, M) = 0$ and so $\operatorname{Hom}_\Delta(\underline{\underline{U}}_{-n+1}, M\text{-}p) \hookrightarrow \operatorname{Ext}^1_\Delta(\underline{\underline{U}}_{-n+1}, HW(M))$ and as $HW(M)$ is Hodge-Witt $\dim_k \operatorname{Ext}^1_\Delta(\underline{\underline{U}}_{-n+1}, HW(M)) = m^{1,-1}(HW(M)) = m^{1,-1}(M)$ and so $2n-1 \leq m^{1,-1}(M) \leq \max_i(m^{i,-i}(M))$.

Let us finally specify the behaviour of Δ_{MO} under internal tensor operations.

Proposition 4.11. Δ_{MO} *is stable under* $(-)\hat{*}^L_R(-)$ *and* $\mathrm{RHom}^!_R(-,-)$.

Indeed, it is easy to see (cf. (IV:2.2)) that $M \in D^b_c(R)$ is in Δ_{MO} iff $H^i(R_1 \otimes^L_R M)^j = 0$ whenever $i+j \neq 0$. Using this and $R_1 \otimes^L_R ((-)\hat{*}^L_R(-)) = R_1 \otimes^L_R(-) \otimes_k R_1 \otimes^L_R (-)$ and $R_1 \otimes^L_R \mathrm{RHom}^!_R(-,-) = \mathrm{Hom}_k(R_1 \otimes^L_R (-), R_1 \otimes^L_R (-))$ (cf. (0:3.3;3.3.1)) immediately gives the result.

IV

Coherent R-complexes

<u>Definition</u> 1.1. *Let* $M \in D_c^b(R)$. *Then* M *is Mazur-Ogus in degree* n *iff* $b_n(M) = \sum_{i+j=n} h^{i,j}(M)$ *and* M *is Mazur-Ogus if* M *is Mazur-Ogus in degree* n *for all* n *i.e.* $\sum_n b_n(M) = \sum_{i,j} h^{i,j}(M)$ *(as* $b_n(M) \geq \sum_{i+j=n} h^{i,j}(M)$ *always).*

<u>Theorem</u> 1.2. i) $M \in D_c^b(R)$ *is Mazur-Ogus in degree* n *iff* $\widetilde{H}^n(M)$ *is Mazur-Ogus,* $G^2(\widetilde{H}^{n-1}(M)) = \widetilde{H}^{n-1}(M)$ *and* $F^1(\widetilde{H}^{n+1}(M)) = 0$.

ii) *Let* $M \in D_c^b(R)$ *be Mazur-Ogus in degree* n . *Then the Hodge filtration on* $H^n(\underline{\underline{s}}(M))/p$ *equals the reduction modulo* p *of the Hodge filtration on the virtual* F*-crystal* $H^n(\underline{\underline{s}}(M)) \otimes K$.

iii) *Let* $M \in D_c^b(R)$ *be Mazur-Ogus in degree* n. *Then the filtrations in* $H^n(\underline{\underline{s}}(M)) \otimes K$ *and* $H^{n+1}(\underline{\underline{s}}(M)) \otimes K$ *coming from the s.s.* $H^i(M)^j \Longrightarrow H^{i+j}(\underline{\underline{s}}(M))$ *equals the filtration by maximal virtual* F*-crystals divisible by successive powers of* p .

iv) $(H=1)$ *Let* $M \in D_c^b(R)$ *be Mazur-Ogus in degree* n . *Then* $M \simeq \widetilde{\tau}_{\leq n-1} M \oplus \widetilde{\tau}_{>n-1} M$. *In particular if* M *is Mazur-Ogus then* $M \simeq \bigoplus_n t'(\mathrm{Hodge}(H^n(\underline{\underline{s}}(M)) \otimes K))[-n]$.

v) *Suppose* $M \in D_c^b(R)$ *is Mazur-Ogus in degree* m *but not Hodge-Witt in degree* m . *Suppose also that* $nb_m(M) = \sum_{i+j=m} \mathrm{lgth}\, H^i(R_n \otimes_R^L M)^j$ *for some* n . *Then* $2n-1 \leq \max_i m^{m+i,-i}$.

Proof : We have that $b_n(M) = b_0(\widetilde{H}^n(M))$ and by (I:1.4.1 ii) $h^{i,n-i}(M) \geq h^{i,-i}(\widetilde{H}^n(M))$ for all i. As $b_0(\widetilde{H}^n(M)) \geq \sum_i h^{i,-i}(\widetilde{H}^n(M))$ the condition of i) gives $b_0(\widetilde{H}^n(M)) = \sum_i h^{i,-i}(\widetilde{H}^n(M))$ and so $\widetilde{H}^n(M)$ is Mazur-Ogus by (III:4.1). Also $h^{i,-i}(\widetilde{H}^n(M)) = h^{i,n-i}(M)$ and as $\widetilde{H}^n(M)$ is MO $h^{i,j}(\widetilde{H}^n(M)) = 0$ if $i+j \neq 0$. The s.s.

$$H^i(R_1 \otimes^L_R \widetilde{H}^j(M)) \Longrightarrow H^{i+j}(R_1 \otimes^L_R M) \tag{1.2.1}$$

and (I:1.4 ii) then gives $h^{i,j}(\widetilde{H}^{n-1}(M)) = 0$ if $i+j = 1$ and $h^{i,j}(\widetilde{H}^{n+1}(M)) = 0$ if $i+j = -1$. This is equivalent to $G^2(\widetilde{H}^{n-1}(M)) = \widetilde{H}^{n-1}(M)$ and $F^1(\widetilde{H}^{n+1}(M)) = 0$ (cf. (III:Lemma 2.2)).

Conversely, if $\widetilde{H}^n(M)$ is Mazur-Ogus, $G^2(\widetilde{H}^{n-1}(M)) = \widetilde{H}^{n-1}(M)$ and $F^1(\widetilde{H}^{n+1}(M)) = 0$ then $h^{i,j}(\widetilde{H}^{n-1}(M)) = 0$ if $i+j = 1$ and $h^{i,j}(\widetilde{H}^{n+1}(M)) = 0$ if $i+j = -1$ and so (1.2.1) shows that $h^{i,-i}(\widetilde{H}^n(M)) = h^{i,n-i}(M)$ and thus $b_n(M) = b_0(\widetilde{H}^n(M)) = \sum_i h^{i,-i}(\widetilde{H}^n(M)) = \sum_{i+j=n} h^{i,j}(M)$. As for ii), (III:4.3) and (II:3.6 iii) shows that the result is true for $\widetilde{H}^n(M)$ as $\widetilde{H}^n(M)$ is Mazur-Ogus. By i) $H^n(\underline{\underline{s}}(M)) \otimes K = H^0(\underline{\underline{s}}(\widetilde{H}^n(M))) \otimes K$ as filtered virtual F-crystals (i.e. $\widetilde{H}^{n-1}(M) \in G$) and we have seen that $\tau_{\leq n}M \longrightarrow M$ and $\tau_{\leq n}M \longrightarrow \widetilde{H}^n(M)[-n]$ induce isomorphisms when $H^i(R_1 \otimes^L_R (-))^j$ $i+j = n$ is applied. Thus the Hodge filtrations on $H^n(\underline{\underline{s}}(M))/p$ and $H^0(\underline{\underline{s}}(\widetilde{H}^n(M))) \otimes K$ coincide and the result is proven.

In iii) the arguments of the proof of (I:2.3) allows us to reduce to $n = 0$ and $M \in \Delta$ and, by i), $F^1M = 0$. The inclusion $F^2M \hookrightarrow M$ induces an isomorphism on $H^0(\underline{\underline{s}}(-))$ and as the inclusion of the E_∞-filtration in the p-adic divisibility filtration is clear we reduce by functoriality to F^2M which as $F^1M = 0$ is Mazur-Ogus and we finish by (III:4.5).

The obstruction to decomposing M as $\widetilde{\tau}_{\leq n-1}M \oplus \widetilde{\tau}_{>n-1}M$ is an element of $\mathrm{Ext}^1_{D(R)}(\widetilde{\tau}_{>n-1}M, \widetilde{\tau}_{\leq n-1}M)$ so it will be enough to show that this group is 0.

Now $\mathrm{RHom}_R(-,-)^0 = \mathrm{RHom}_R(W, \mathrm{RHom}^!_R(-,-))^0 = (\mathrm{RHom}^!_R(-,-)^F)^0$. As $((-)^F)^0$ has Δ-amplitude $[0,1]$ (dévissage and check) it will be enough to show that $\mathrm{RHom}^!_R(\tilde{\tau}_{>n-1}M, \tilde{\tau}_{\leq n-1}M)$ has Δ-amplitude $]-\infty,-1]$ and for this it will be enough to show that it has G-amplitude $]-\infty,-1]$. By i) and (I:4.1) we get that $(R_1 \otimes^L_R \tilde{\tau}_{>n-1}M)^j$ has amplitude $[n+j,\infty[$ and $\underline{s}(\tilde{\tau}_{>n-1}M)$ $[n,\infty[$ and by i) and (II:1.2) $(R_1 \otimes^L_R \tilde{\tau}_{\leq n-1}M)^j$ has amplitude $]-\infty,n-1+j]$ and $\underline{s}(\tilde{\tau}_{\leq n-1}M)$ $]-\infty,n-1]$. We now conclude using (II:1.2) as in (II:1.2.2).

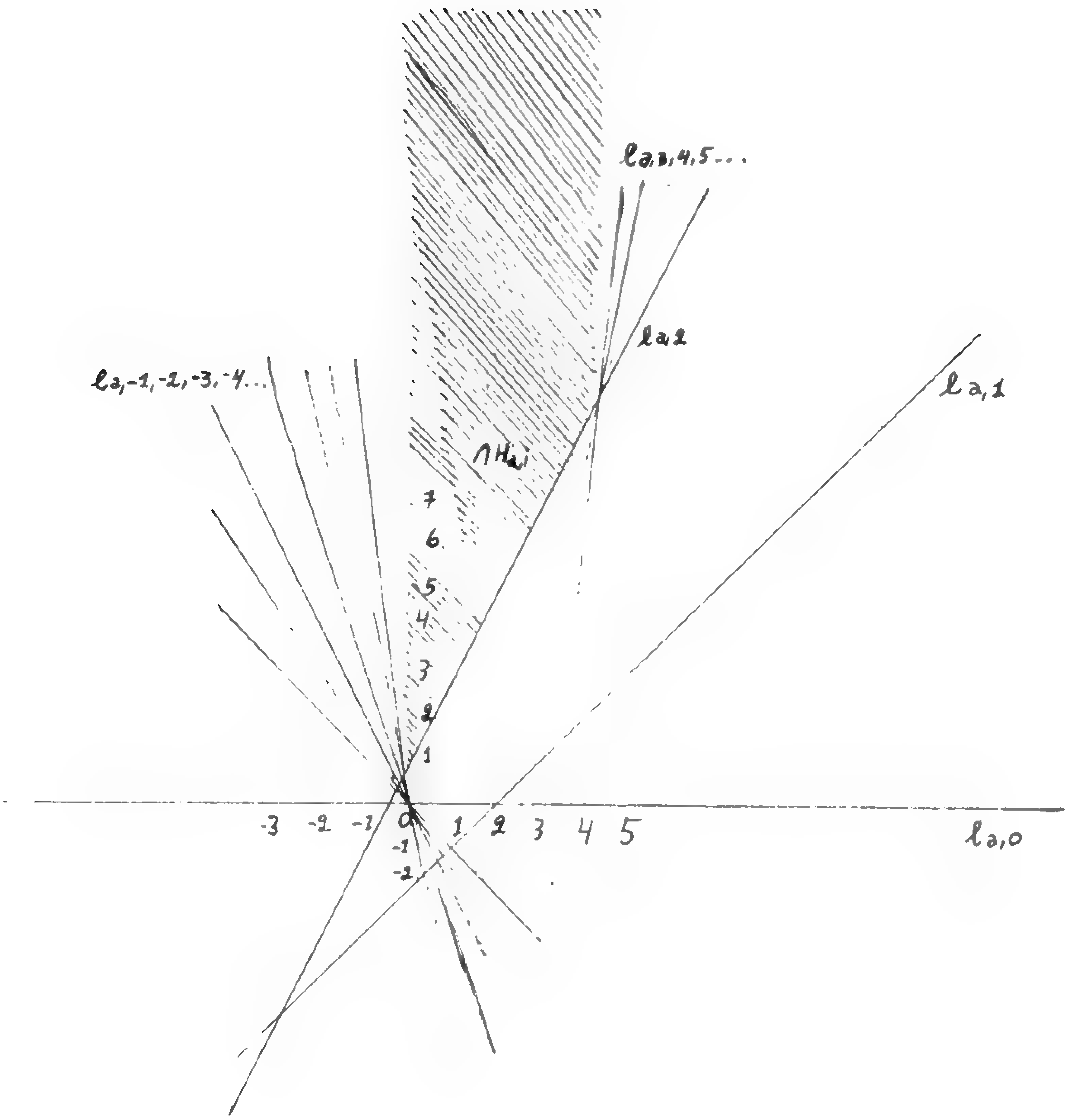
As for v) we argue as before and conclude that $nb_0(\tilde{H}^m(M)) = \sum_i H^i(R_n \otimes^L_R \tilde{H}^m(M))^{-i}$ so that the condition of (III:4.10.1 ii) is fulfilled and we conclude by the result obtained there.

The Hodge-Witt numbers

2. Let $a = (\ldots, a^{-2}, a^{-1}, a^0, a^{-2}, \ldots)$ be a sequence of real numbers almost all of which are zero. For each $i \in \mathbb{Z}$ consider the closed half plane $\subseteq \mathbb{R}^2$, $H_{a,i}$, which is bounded by the line $\ell_{a,i}$ of slope i passing through the point $(\sum_{j\leq i} a^j, \sum_{j\leq i} ja^j)$ and which contains an unbounded part of the positive y-axis.

Definition 2.1. *The Hodge polygon,* Hodge(a), *of* a *is the boundary of* $\bigcap_i H_{a,i}$.

Example 2.1.1. Let $a^0 = 2$, $a^1 = -5$, $a^2 = 7$ and the rest 0 . Then we have the following picture



Clearly Hodge(a) will in general be a finite sided polygon with two unbounded sides, one which is a part of the positive part of the y-axis and the other paralell to and to the right of the y-axis. It is easy to see that to a there is a sequence b with the same properties as a and furthermore that the b^i are non-negative except possibly for the first non-zero element b^j and that if this $b^j < 0$ and $j < 0$ then $b^j = -b^{j+1}$, s.t. Hodge(a) = Hodge(b) and that such a b is unique.

Remark : All $b^i \geq 0$ iff the whole positive y-axis belongs to Hodge(a).

Example : In (2.1.1) $b^0 = -1/2$, $b^2 = 1/2$ and all the other $b^i = 0$.

Definition 2.2. *A sequence* $a = (\ldots, a^{-2}, a^{-1}, a^0, a^1, a^2, \ldots)$ *with almost all* $a^i = 0$ *is normalized if all* $a^i \geq 0$ *except possibly the first non-zero element* a^j *and that* $a^j = -a^{j+1}$ *if* $a^j < 0$ *and* $j < 0$. *If* a *is any sequence with almost all* $a^i = 0$ *then the unique normalized* b *with* Hodge(a) = Hodge(b) *is called the associated normalized sequence.*

Definition 2.3. *If* $a = (a^i)_{i \in \mathbb{Z}}$ *is a sequence of real numbers with* $a^i = 0$ *for* $i \ll 0$ *then we define new sequences* Σa *and* Δa *by* $(\Delta a)^i = a^i - a^{i-1}$ *and* $(\Sigma a)^i = \sum_{j \leq i} a^i$. *Then* Σ *and* Δ *are linear endomorphisms of the space of such sequences, inverses to each other, and* Σ *is increasing (under the pointwise partial ordering of sequences). We put*

$$\Sigma^2 = \Sigma \circ \Sigma \qquad \Delta^2 = \Delta \circ \Delta \ .$$

If a and b are two sequences with almost all the a^i and b^i zero then we say that Hodge(a) is above Hodge(b) (Hodge(a) $\geq$ Hodge(b)) if Hodge(a) $\subseteq \bigcap_i H_{b,i}$.

Proposition 2.4. i) *If* a *and* b *are sequences with finite support s.t.* $\Sigma^2 b \geq \Sigma^2 a$ *(in the pointwise ordering) then* Hodge(a) $\geq$ Hodge(b) *and conversely if* $a \geq 0$.

ii) *Let* $\Sigma^2 b \geq \Sigma^2 a$. *If* $(\Sigma^2 a)^i = (\Sigma^2 b)^i$ *and* a *is normalized then the slope* i+1 *part of* Hodge(a) *is contained in the slope* i+1 *part of* Hodge(b) *and conversely if* b *is also normalized.*

iii) *If* $(\Sigma^2 a)^i = (\Sigma^2 b)^i$ *for all* $i \gg 0$ *then* Hodge(a) *and* Hodge(b) *have the same endpoints where the endpoint of a Hodge polygon is the vertex lying on the rightmost vertical side and conversely if* a *and* b *are normalized.*

To prove this let us first note that $\ell_{a,i+1}$ meets the y-axis in $(0,-(\Sigma^2 a)^i)$ and similarly for $\ell_{b,i+1}$. This is easy linear algebra. Hence $(\Sigma^2 b)^i \geq (\Sigma^2 a)^i$ iff $H_{a,i+1} \subseteq H_{b,i+1}$ and $\ell_{a,i+1} = \ell_{b,i+1}$ iff $(\Sigma^2 a)^i = (\Sigma^2 b)^i$. If therefore $\Sigma^2 b \geq \Sigma^2 a$ then $\forall i : H_{a,i} \subseteq H_{b,i}$ so $\text{Hodge}(a) \subseteq \bigcap_i H_{a,i} \subseteq \bigcap_i H_{b,i}$ and, by definition, $\text{Hodge}(a) \geq \text{Hodge}(b)$. As for the converse, if $\ell_{a,i+1} \cap \text{Hodge}(a) \neq \emptyset$ then $\ell_{a,i+1} \cap \bigcap_j H_{b,j} \neq \emptyset$ and so $\ell_{a,i+1} \subseteq H_{b,i+1}$ which gives $H_{a,i+1} \subseteq H_{b,i+1}$ and $(\Sigma^2 b)^i \geq (\Sigma^2 a)^i$. However as $a \geq 0$ it is clear that $\ell_{a,i} \cap \text{Hodge}(a) \neq \emptyset$ for all i. (Note that $a >> 0$ is equivalent to $(0,0) \in \text{Hodge}(a)$ and that a is normalized).

In ii) as the slope $i+1$ part of $\text{Hodge}(a)$ is $\ell_{a,i+1} \cap \text{Hodge}(a)$ and by assumption $\text{Hodge}(a) \geq \text{Hodge}(b)$ and $\ell_{a,i+1} = \ell_{b,i+1}$ we get either $\ell_{a,i+1} \cap \text{Hodge}(a) = \emptyset$ or $\ell_{b,i+1} \cap \text{Hodge}(b) \neq \emptyset$ and then it is the slope $i+1$ part of $\text{Hodge}(b)$ and $\ell_{a,i+1} \cap \text{Hodge}(a) \subseteq \ell_{b,i+1} \cap \text{Hodge}(b)$ and in either case we get what we want. The converse is also clear.

As for iii), the endpoint is equal to the slope i part for all $i >> 0$ and we conclude by ii).

We will need two more concepts. If P is a finite sided convex polygon P beginning with $\{(0,y) : a \leq y\}$ for some $a \geq 0$ and ending with $\{(b,y)$ $c \leq y\}$ for some $b,c \geq 0$ then there is a unique normalized sequence a s.t. $P \subseteq \bigcap_i H_{a,i}$ and $\text{Hodge}(a) \geq \text{Hodge}(b)$ for all b with $P \subseteq \bigcap_i H_{b,i}$. Simply let $\ell_{a,i}$ be the line of slope i which touches P but does not cut it. Note that $\ell_{a,i}$ will touch P where P changes slopes from $< i$ to $\geq i$. We will say that $\text{Hodge}(a)$ is the highest Hodge polygon below P.

If $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \leq \lambda_s$ is an increasing sequence of real numbers we define its Newton polygon as follows. We start with the positive y-axis, then the line segment from $(0,0)$ to $(1,\lambda_1)$, then the line segment from $(1,\lambda_1)$ to $(2,\lambda_1+\lambda_2)$ etc. and we finish by $\{(s,y) : y \geq \lambda_1+\lambda_2+\ldots+\lambda_s\}$.

Proposition 2.5. *Let* $\lambda_1 \le \lambda_2 \le \lambda_3 \dots \le \lambda_s$ *be an increasing sequence of real numbers. The highest Hodge polygon below the Newton polygon of this sequence has as normalized sequence* m *with*

$$m^i = \sum_{\lambda_r \in [i,i+1[} (1+i-\lambda_r) + \sum_{\lambda_r \in [i-1,i[} (\lambda_r - i + 1) \ .$$

The proof is an easy exercise in linear algebra using that $\ell_{m,i}$ is touching the Newton polygon where it changes slopes from $< i$ to $\geq i$.

3. Let us now define the Hodge-Witt numbers.

Definition 3.1. *Let* $M \in D_c^b(R)$. *Define for each* $n \in \mathbb{Z}$ *sequences* $T^{\cdot,n-\cdot}(M)$, $m^{\cdot,n-\cdot}(M)$, $h^{\cdot,n-\cdot}(M)$ *and* $h_W^{\cdot,n-\cdot}(M)$ *(Hodge-Witt numbers) by*

$$(T^{\cdot,n-\cdot})^i := T^{i,n-i}(M)$$

$$(m^{\cdot,n-\cdot})^i := m^{i,n-i}(M)$$

$$(h^{\cdot,n-\cdot})^i := h^{i,n-i}(M)$$

$$h_W^{\cdot,n-\cdot} := m^{\cdot,n-\cdot} + \Delta^2(T^{\cdot,n-\cdot}) \ .$$

Put $h_W^{i,n-i} = (h_W^{\cdot,n-\cdot})^i$.

Theorem 3.2. *Let* $M \in D_c^b(R)$.

i) $b_n(M) = \sum_{i+j=n} h_W^{i,j}(M)$ *for all* n .

ii) *(Crew's formula)* $\sum_j (-1)^j h^{i,j}(M) = \sum_j (-j)^j h_W^{i,j}(M)$ *for all* i .

Proof : As $h^{\cdot,n-\cdot}$ has finite support $\sum_{i+j=n} h_W^{i,j} = (\Sigma h^{\cdot,n-\cdot})^k$ for $k \gg 0$.

Hence for $k \gg 0$ $\sum_{i+j=n} h_W^{i,j} = (\Sigma m^{\cdot,n-\cdot} + \Sigma\Delta^2 T^{\cdot,n-\cdot})^k = (\Sigma m^{\cdot,n-\cdot})^k + (\Delta T^{\cdot,n-\cdot})^k =$

$\sum_{i+j=n} m^{i,j}$, the last equality as $m^{\cdot,n-\cdot}$ and $T^{\cdot,n-\cdot}$ have finite support

but $\sum_{i+j=n} m^{i,j} = b_n$ by (0:6.2). In ii) the right hand side is additive on distinguished triangles and one verifies easily that so is the left hand side (cf. (III:2.6)). We reduce then to elementary R-modules and compute.

Remark: In his thesis Crew, [Cr], proved a formula which up to a rearrangement of the terms is 3.2 ii). His proof is the same as the one given here. Another rearrangement has been proved, independently, by Milne (to appear) by a slightly different method.

Theorem 3.3. *Let* $M \in D^b_c(R)$. *Then* $h_W^{i,j}(M) \leq h^{i,j}(M)$ *for all* i,j.

By shifting we may assume $i+j=0$. We then have $h^{i,j}(M) \geq h^{i,j}(\tilde{H}^0(M))$ by (I:Cor. 1.4.1) and evidently $h_W^{i,j}(M) = h_W^{i,j}(\tilde{H}^0(M))$, so we may assume $M \in \Delta$. By (I:1.4) $h^{i,j}(M) = 0$ if $|i+j| > 1$ and clearly $h_W^{i,j}(M) = 0$ if $i+j \neq 0$ (as this is true of $m^{i,j}$ and $T^{i,j}$). Hence by (3.2 ii) $h_W^{i,j} = h^{i,j} - (h^{i,j-1} + h^{i,j+1}) \leq h^{i,j}$.

Corollary 3.3.1. *Suppose that* $M \in D^b_c(R)$ *is Mazur–Ogus in degree* n. *Then* $h_W^{i,j}(M) = h^{i,j}(M)$ *for* $i+j=n$ *and so* $T^{\cdot,n-\cdot} = \Sigma^2(h^{\cdot,n-\cdot} - m^{\cdot,n-\cdot})$.

Proof : By (3.2 i), (3.3) and assumption we have $b_n = \sum_{i+j=n} h_W^{i,j} \leq \sum_{i+j=n} h^{i,j} = b_n$, whence $h_W^{i,j} = h^{i,j}$, $i+j=n$.

Definition 3.4. *Let* $M \in D^b_c(R)$.

i) *The Newton polygon in degree* n, $\mathrm{Newton}_n(M)$ *of* M *is the Newton polygon of the slopes of the virtual* F*-crystal* $H^n(\underline{\underline{s}}(M)) \otimes K$.

ii) *The Newton–Hodge polygon in degree* n, $\mathrm{Newton\text{-}Hodge}_n(M)$, *of* M *is the highest Hodge polygon below* $\mathrm{Newton}_n(M)$.

iii) *The abstract Hodge polygon in degree* n, $\mathrm{Hodge}_n^{abs}(M)$, *of* M *is the Hodge polygon of the Hodge numbers of the virtual* F*-crystal* $H^n(\underline{\underline{s}}(M) \otimes K)$, *where these Hodge numbers are defined with the aid of the elementary*

divisors of the Frobenius (cf. [Ka]).

iv) *The Hodge-Witt polygon in degree* n , $\mathrm{Hodge\text{-}Witt}_n(M)$, *of* M *is* $\mathrm{Hodge}(h_W^{\cdot,n-\cdot})$.

v) *The Hodge polygon in degree* n , $\mathrm{Hodge}_n(M)$, *of* M *is* $\mathrm{Hodge}(h^{\cdot,n-\cdot})$.

<u>Theorem</u> 3.5. *Let* $M \in D_c^b(R)$. *Then, for all* n ,

$$\mathrm{Newton}_n(M) \geq \mathrm{Newton\text{-}Hodge}_n(M) \geq \mathrm{Hodge}_n^{abs}(M) \geq \mathrm{Hodge\text{-}Witt}_n(M) \geq \mathrm{Hodge}_n(M)$$

and all but $\mathrm{Hodge}_n(M)$ *have the same endpoints.*

Indeed, by shifting we may assume $n = 0$. $\mathrm{Newton}_0(M) \geq \mathrm{Newton\text{-}Hodge}_0(M)$ and by (3.3) we have $h_W^{i,-i} \leq h^{i,-i}$ and so, <u>a fortiori</u> $\mathrm{Hodge\text{-}Witt}_0(M) \geq \mathrm{Hodge}_0(M)$. By (2.5) and (0:6.2) $\mathrm{Newton\text{-}Hodge}_0(M) = \mathrm{Hodge}(m^{\cdot,0-\cdot})$ and $\Sigma^2(h_W^{\cdot,0-\cdot}) = \Sigma^2(m^{\cdot,0-\cdot}) + T^{\cdot,0-\cdot} \geq \Sigma^2(m^{\cdot,0-\cdot})$ and so, by (2.4), $\mathrm{Newton\text{-}Hodge}_0 \geq \mathrm{Hodge\text{-}Witt}_0$. Consider now $N = F^2\widetilde{H}^0(M)/F^1\widetilde{H}^0(M)$. By (I:4.5.3) and (I:Cor. 1.4.1) the virtual F-crystals $H^0(\underline{\underline{s}}(M)) \otimes K$ and $H^0(\underline{\underline{s}}(F^2\widetilde{H}^0(M)/F^1\widetilde{H}^0(M))) \otimes K$ are equal and so have the same abstract Hodge polygons. On the other hand N is Mazur-Ogus so by (1.2 iii) $\mathrm{Hodge}_0(N) = \mathrm{Hodge}_0^{abs}(N)$ and by (3.3) $\mathrm{Hodge\text{-}Witt}_0(N) = \mathrm{Hodge}_0(N)$. Finally, by (III:2.6) $T^{\cdot,0-\cdot}(N) \leq T^{\cdot,0-\cdot}(M)$ and as $m^{\cdot,0-\cdot}(N) = m^{\cdot,0-\cdot}(M)$ we get $\mathrm{Hodge\text{-}Witt}_0(N) \geq \mathrm{Hodge\text{-}Witt}_0(M)$ by (2.4) which gives $\mathrm{Hodge}_0^{abs}(M) = \mathrm{Hodge}_0^{abs}(N) \geq \mathrm{Hodge\text{-}Witt}_0(M)$.

For the endpoints it is now sufficient to prove that $\mathrm{Newton\text{-}Hodge}_n(M)$ and $\mathrm{Hodge\text{-}Witt}_n(M)$ have the same endpoints. This follows from (2.4 iii) as $T^{\cdot,n-\cdot}(M)$ has finite support.

<u>Proposition</u> 3.6. *Let* $M \in D_c^b(R)$. *Then* $\mathrm{Newton}_n(M)$ *touches the slope* i *part of* $\mathrm{Hodge}_n M$ *iff* $\widetilde{H}^n(M)$ *is cut at* $\{i-1\}$ *and* $h_W^{j,n-j}(M) = h^{j,n-j}(M)$ *for* $j \leq i-1$.

Indeed, as $m^{\cdot,n-\cdot}$ and $h^{\cdot,n-\cdot}$ are both normalized (2.4 ii) shows that $\mathrm{Newton}_n M$ touches $\mathrm{Hodge}_n M$ in slope i iff $(\Sigma^2 m^{\cdot,n-\cdot})^{i-1} = (\Sigma^2 h^{\cdot,n-\cdot})^{i-1}$. But $\Sigma^2 h^{\cdot,n-\cdot} = T^{\cdot,n-\cdot} + \Sigma^2(h^{\cdot,n-\cdot} - h_W^{\cdot,n-\cdot}) + \Sigma^2 m^{\cdot,n-\cdot}$ so by (3.3) $(\Sigma^2 m^{\cdot,n-\cdot})^{i-1} = (\Sigma^2 h^{\cdot,n-\cdot})^{i-1}$ iff $T^{i-1,n-i+1} = 0$ and $h_W^{j,n-j} = h^{j,n-j}$ $j < i$. Now $\widetilde{H}^n(M)$ is cut at $\{i-1\}$ iff $T^{i-1,n-i+1} = 0$.

Corollary 3.6.1. *Let* $M \in D_c^b(R)$. *Then* M *is Mazur-Ogus and Hodge-Witt in degree* n *iff* $\mathrm{Newton}_n M$ *touches the slope* i *part of* $\mathrm{Hodge}_n M$ *for all* i.

Proof : As $b_n(M) = \sum_{i+j=n} h^{i,j} + \sum_{i+j=n} (h_W^{i,j} - h^{i,j})$ we see by (3.3) that M is Mazur-Ogus in degree n iff $h_W^{i,j} = h^{i,j}$ for $i+j = n$.

Remark. This is the result alluded to in [Ka 1].

Duality

4. Let us recall, (0:3), that $D(-) := R\mathrm{hom}_R^!(-,W)$. In [Ek 1] I have proved that $D(-)$ has amplitude [0,2] and the reader will there find a complete description of its effect on various types of coherent R-modules. In I we proved that $D(-)$ has (Δ,Δ)-amplitude (cf. (0:1)) [0,1] and the diagonal complexes of form $\widetilde{H}^0(D(M))$ resp. $\widetilde{H}^1(D(M))$ for $M \in \Delta$ are exactly the diagonal complexes without finite torsion resp. those that are finite torsion.

Proposition 4.1. i) $D(-)$ *has* (G,G)-*amplitude* [0,2].

ii) $D_g^i(D_g^j(-)) = 0$ *on* G *unless* $i = j$ *or* $(i,j) = (2,1)$, *where* $D_g^i(-) := H_g^i \circ D(-)$.

iii) *Let* $M \in G$. $D_g^0(D_g^0(M)) = t'(\mathrm{Hodge}(\underline{\underline{s}}(M)/\mathrm{torsion}))$, $D_g^2(D_g^2(M)) =$ *the maximal subobject of* M *whose associated simple complex is zero,* $D_g^1(D_g^1(M)) = \mathrm{torsion}(M)/D_g^2(D_g^2(M))$ *and* $D_g^2(D_g^1(M)) = D_g^0(D_g^0(M))/(M/\mathrm{torsion})$.

Proof : We get i) as a special case of (II:1.2.2 iii). Let for $M \in G$, M_1 be the maximal subobject with $\underline{\underline{s}}(-) = 0$ and $M_2 := \mathrm{tors}(M)$. I claim that

$h^{i,j}(M_2/M_1)=0$ unless $i+j=0$ or -1. Indeed, if not then by (II:1.2) $h^{i,j}(M_2/M_1)\neq 0$ for some (i,j) with $i+j=-2$ and so by (0:4.3) there is a non-zero $\underline{\underline{U}}_0[i+1](-i) \longrightarrow M_2/M_1$ for some i. As $\underline{\underline{U}}_0[i+1](-i)$ is simple as object of G (e.g. as its associated F-gauge structure visibly is) this morphism is a monomorphisme so M_2/M_1 would contain a non-zero subobject with $\underline{\underline{s}}(-)=0$ contrary to definition of M_1. The same argument shows that $h^{i,j}(M/M_1)=0$ unless $i+j=0$ or -1. Using (0:3.3) and (II:1.2) we see that $D(M/M_1)$ has G-amplitude $[0,1]$ so $D^2_g(M)=D^2_g(M_1)$ on the other hand $\underline{\underline{s}}(M_1)=0$ and $h^{i,j}(M_1)=0$ unless $i+j=0,-1$ or -2 so by (0:3.3) $D(M_1)$ has amplitude $[2,2]$ so $D^0_g(M)=D^0_g(M/M_1)$ and $D^1_g(M)=D^1_g(M/M_1)$. I claim that $D(M_2/M_1)=D^1_g(M_2/M_1)[-1]$. In fact as for M/M_1 we see that $D^2_g(M_2/M_1)=0$. Now as $h^{i,j}(M_2/M_1)=0$ unless $i+j=0$ or -1 and $\underline{\underline{s}}(M_2/M_1)$ is torsion as M_2/M_1 is we see by (0:3.3) that $h^{i,j}(D(M_2/M_1))=0$ unless $i+j=0$ or 1 and $\underline{\underline{s}}(D(M_2/M_1))$ is concentrated in degree 1. Hence $\underline{\underline{s}}(D^0_g(M_2/M_1))=H^0(\underline{\underline{s}}(D(M_2/M_1))=0$ and $h^{i,-i-2}(D^0_g(M_2/M_1))=h^{i,-i-2}(D(M_2/M_1))=0$. However, $\underline{\underline{s}}(D^0_g(M_2/M_1))=0$ so $D^0_g(M_2/M_1)[-1]$ is an $\underline{\underline{s}}$-acyclic diagonal complex and so $D^0_g(M_2/M_1)$ if non-zero contains some $\underline{\underline{U}}_0[i+1](-i)$ as subobject which contradicts $h^{i+1,-i-3}(D^0_g(M_2/M_1))=0$ by (0:4.3). In the same way as $h^{i,-i-2}(D^1_g(M_2/M_1))=h^{i,-i-1}(D(M))=0$ we see that $D^1_g(M^2/M^1)$ contains no non-zero object with $\underline{\underline{s}}(-)=0$ and as it is torsion as M^2/M^1 is by what we have just shown $D^0_g(D^1_g(M^2/M^1))=D^2_g(D^1_g(M^2/M^1))=0$. As for $D^2_g(M_1)=D(M_1)$ we see that it has $\underline{\underline{s}}(-)=0$ and so, again by what has been shown, $D^0_g(D^2_g(M_1))=D^1_g(D^2_g(M_1))=0$ and as $D^2_g(M_1)=D^2_g(M)$ the same is true for M. Further M/M_2 is without torsion so it embeds in $M'=t'(\mathrm{Hodge}(\underline{\underline{s}}(M/M_2))$ with a quotient N that has $\underline{\underline{s}}(-)=0$. As M' is in Δ_{M0}, $D(M')=\widetilde{D}^0(M')$ by (III:4.11) and $D^0(M')$ is again in Δ_{M0} and so in G. Thus $D(M')=D^0_g(M')$. The long

exact sequence $0 \to D_g^0(N) \to D_g^0(M') \to D_g^0(M/M_2) \to \dots \to D_g^2(M/M_2) \to 0$ and the fact that $D_g^0(N) = D_g^1(N) = 0 = D_g^2(M/M_2) = D_g^2(M') = D_g^1(M')$ give $D_g^0(M') = D_g^0(M/M_2)$ and $D_g^1(M/M_2) = D_g^2(N)$. As $D_g^0(M')$ is in Δ_{M0} $D_g^1(D^0(M')) = D_g^2(D^0(M')) = 0$. To summarize we have obtained $D_g^0(M) = D_g^0(M')$, an exact sequence $0 \to D_g^2(N) \to D_g^1(M) \to D_g^1(M_2/M_1) \to 0$ and $D_g^2(M) = D_g^2(M_2)$. Hence, $D_g^1(D_g^0(M)) = D_g^2(D_g^0(M)) = D_g^0(D_g^1(M)) = D_g^0(D_g^2(M)) = D_g^1(D_g^2(M)) = 0$, $D_g^2(D_g^1(M)) = D_g^2(D_g^1(N)) = D(D(N))$, $D_g^0(D_g^0(M)) = D_g^0(D_g^0(M')) = D(D(M'))$, $D_g^1(D_g^1(M)) = D_g^1(D_g^1(M_2/M_1)) = D(D(M_2/M_1))$ and $D_g^2(D_g^2(M)) = D_g^2(D_g^2(M_1)) = D(D(M_1))$. As we have $D(D(-)) = id$ we are finished.

Corollary 4.1.1. *The coherent* F-*gauge structures* N *with the properties that they are torsion and that for no non-zero element of* N^i *are the images in* N^∞ *and* $N^{-\infty}$ *zero are exactly those of the form* $\underline{\underline{S}}(D_g^1(M))$ *for a torsion element of* G .

Indeed, it is clear from (4.1) and its proof that the elements of G of the form $D_g^1(M)$ for M torsion are exactly the torsion elements of G containing no non-zero subobject with $\underline{\underline{s}}(-) = 0$. It is also clear that an $M \in G$ contains no non-zero subobject with $\underline{\underline{s}}(-) = 0$ iff $\mathrm{Hom}_G(\underline{\underline{U}}_0^i[1],M) = 0$ for all i . However, $\mathrm{Hom}_G(\underline{\underline{U}}_0^i[1],M) = \mathrm{Hom}_{F\text{-}g\text{-}str}(\underline{\underline{S}}(\underline{\underline{U}}_0^i[1]),\underline{\underline{S}}(M))$ as $\underline{\underline{S}}$ is an equivalence of categories and as $\underline{\underline{S}}(\underline{\underline{U}}_0^i[1])^j = k$ if $i = j$ and 0 if not, we see that $\mathrm{Hom}_{F\text{-}g\text{-}str}(\underline{\underline{S}}(U_0^i[1]),N) = \{x \in N^i : \tilde{F}x = \tilde{V}x = 0\}$. It is clear however that the condition that there be no such non-zero x for all i is equivalent to the second condition of the corollary.

Remark : i) The coherent F-gauge structures fulfilling the condition of the corollary are precisely those F-gauge structures considered by Fontaine and Messing (to appear). That they have the form of the corollary suggests that in their theory the dual point of view could be more useful.

ii) It is no doubt possible to define directly on F-gauge structures a tensor product and internal Hom-functor which would correspond to $(-)\hat{*}^{L}_{R}(-)$ resp. $\mathrm{RHom}^{!}_{R}(-,-)$ through $\underline{\underline{S}}(-)$ but I have not tried to do that.

V

F-Crystals

1. If we apply (IV:3.6) to $t'(\mathrm{Hodge}(M))$ where M is a virtual F-crystal we get the Newton-Hodge decomposition of [Ka].

Theorem 1.1. *Let* M *be a virtual* F-*crystal. Then* $\mathrm{Newton}(M)$ *touches* $\mathrm{Hodge}(M)$ *at slope* i *iff* M *is the sum of its slope* $\leq i$ *part and its slope* $> i$ *part.*

Proof : By (0:5.1) it is clear that M is such a sum iff $t'(\mathrm{Hodge}(M))$ is cut at $\{i\}$.

2. Let us put S the Zariski topos of k , S^{perf} the topos of sheaves on perfect k-schemes with the étale topology (cf.[Be 2]) and $S^{f\ell}$ the topos of sheaves on k-schemes with the fpqc-topology. We ring S by W and $S^{f\ell}$ and S^{perf} by the ring scheme $\underline{W}$. Then we have natural morphisms of ringed topoi $\pi : S^{f\ell} \longrightarrow S$, $\pi : S^{perf} \longrightarrow S$ and $\rho : S^{f\ell} \longrightarrow S^{perf}$ where ρ_* is the restriction. We then have $\pi\rho = \pi$. Note that $L\pi^*(W/p^n) =$ $(\underline{W} \xrightarrow{p^n} \underline{W}) \in D(S^{f\ell}-\underline{W})$ (resp. $D(S^{perf}-\underline{W})$) and so has cohomology $\underline{W}_n$ and ${}_{F^n}\underline{W}$ in degree 0 and -1 (resp. $\underline{W}_n$ in degree 0) and the rest 0 . In particular we see that if M is a finitely generated W-module $L\pi^*M$ is an affine (resp. an affine perfect) group scheme. I claim that for any W-complex M , $R\rho_*L\pi^*M = L\pi^*M$. Indeed we reduce to showing that for any perfect affine k-scheme U and any index set I $H^i_{f\ell}(U, \bigoplus_I \underline{W}) = 0$ for $i > 0$.
As U is quasi-compact we may assume that I contains just one element and then it is clear as U is affine. One also easily checks that for any finite type affine k-group scheme G $R\rho_*G = \rho_*G$ which is simply the perfection of

G. Clearly every σ^n-linear map, for $n \geq 0$, between W-complexes induces naturally a morphism between $L\pi^*$ applied to these complexes. (In the case of S^{perf} we can even allow n to be an arbitrary integer).

If $M = \oplus M^i$ is a complex of F-gauge structures then we define a complex of graded sheaves $M^F \in D(S^{F\ell})$ and a complex of graded inverse systems of sheaves $M^F_{\cdot} \in D(S^{perf})^{\mathbb{N}})$ as follows. We let $(M^F) := \mathrm{Cone}(L\pi^*M^i \xrightarrow{\ell - r} L\pi^*M^{-\infty})$ and $(M^F_{\cdot})^i := \mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} \mathrm{Cone}(L\pi^*M^i \xrightarrow{\ell - r} L\pi^*M^{-\infty})$ where ℓ is induced from the composite of the natural mapping $M^i \longrightarrow M^{\infty}$ and $\tau : M^{-\infty} \longrightarrow M^{\infty}$ and $\mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} N$ is the complex of inverse systems $(\ldots \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \otimes^L_{\mathbb{Z}} N \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \otimes^L_{\mathbb{Z}} N \longrightarrow \ldots)$. By what has been observed before $\mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} R\rho_* M^F = M^F_{\cdot}$ and if M is coherent then the cohomology of M^F (resp. $M^F_{\cdot}$) is an affine group scheme (resp. an inverse system of affine perfect group schemes).

Remark : The reason why we use inverse systems in the case of S^{perf} and not in the case of $S^{f\ell}$ is that an exact sequence of affine group schemes gives an exact sequence of sheaves in $S^{f\ell}$ whereas only exact sequences of finite type affine perfect group schemes give exact sequences in S^{perf}. The reduction modulo successive powers of p will ensure that in the cases of interest all our perfect group schemes will be of finite type. By introducing a topology slightly finer than the étale we could have avoided this reduction.

Recall that for M a complex of R-modules one introduces $M^F_{\cdot}$ a graded inverse system of sheaves in S^{perf} by $M^F_{\cdot} = \mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} \mathrm{Cone}(L\pi^*M \xrightarrow{F-1} L\pi^*M)$. We now have

Proposition 2.1. *Let* $I = [m,n]$ *be a finite interval and* M *a complete* R-*complex of level* I. *Then* $(M^F_{\cdot})^i = ((\underline{\underline{S}}(M))^F_{\cdot})^i[-i]$.

Indeed, $M^F_{\cdot} = \mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} \underline{\underline{s}}(L\pi^*M \xrightarrow{F-1} L\pi^*M)$ which by [Ek 2:I:8.1] equals $\underline{\underline{s}}(R_{\cdot} \otimes^L_R L\pi^*M \xrightarrow{F-1} R_{\cdot} \otimes^L_R L\pi^*M)$. On the other hand $((\underline{\underline{S}}(M))^F_{\cdot})^i = \underline{\underline{s}}(\mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} \underline{\underline{S}}(L\pi^*M)^i \xrightarrow{\ell - r} \mathbb{Z}/p^{\cdot} \otimes^L_{\mathbb{Z}} \underline{\underline{S}}(L\pi^*M)^m)$ where we in the obvious fashion have extended the

notion of F-gauge structures to F-gauge structures on $(S^{perf},\underline{W})$ and $\underline{\underline{S}}$ is the obvious functor from complexes of $R_{S^{perf}}$-modules to complexes of F-gauge structures on S^{perf}. We then see by (the analogue for S^{perf} of) (II:2.3) that we may replace $L\pi^*M$ by its completion so that we want to prove that for a complete $R_{S^{perf}}$-complex N of level I $(\mathbb{Z}/p^{\cdot} \otimes^{L}_{\mathbb{Z}} \underline{\underline{s}}(N \xrightarrow{F-1} N))^{i} = \mathbb{Z}/p^{\cdot} \otimes^{L}_{\mathbb{Z}} \underline{\underline{s}}(\underline{\underline{S}}(N)^{i} \xrightarrow{\ell-r} \underline{\underline{S}}(N)^{m})[-i]$. Now exactly as before one proves that $\underline{\underline{S}}$ is an equivalence of categories from complete $R_{S^{perf}}$-complexes of level I and complete F-gauge structures on S^{perf} of level I. Now, (cf. [Ek 2:III.1.5.4]) $\underline{\underline{s}}(N \xrightarrow{F-1} N)^{i}[i] = \underline{\mathrm{RHom}}_{R_{S^{perf}}}(\underline{W}(i)[-i],N)$ and as $\underline{\underline{S}}$ is an equivalence of categories and $\underline{W}(i)$ and N are complete this equals $\underline{\mathrm{RHom}}_{F\text{-}g\text{-}str}(\underline{\underline{S}}(\underline{W}(i)[-i]),\underline{\underline{S}}(N))$. As the $\hat{G}_r$ are locally projective if we can show that there is an exact sequence $0 \to \hat{G}_m \xrightarrow{F'^{i-m}-V'^{n-i+1}} \hat{G}_i \to \underline{\underline{S}}(\underline{W}(i)[-i]) \to 0$ then we immediately get $\underline{\mathrm{RHom}}_{F\text{-}g\text{-}str}(\underline{\underline{S}}(W(i)),\underline{\underline{S}}(N)) = \underline{\underline{s}}(\underline{\underline{S}}(N)^{i} \xrightarrow{\ell-r} \underline{\underline{S}}(N)^{m})$ which is what we want. This exact sequence is easily established however.

Let now M be a virtual F-crystal and consider $H^1(\mathrm{Hodge}(M)^F) =: M_F$. Being a quotient of a connected and pro-smooth group scheme it is connected and pro-smooth. However, by (2.1) and the fact that $\mathrm{Hodge}(M) \in \Delta$ we see that the perfection of M_F is killed by some power of p and hence so is M_F itself. It is therefore a quotient of a finite sum of π^*M:s divided by some power of p and so a smooth connected unipotent (graded) group scheme.

<u>Definition</u> 2.2. *Let* M *and* N *be virtual* F-*crystals. A morphism* $M \to N$ *is said to be a strong isogeny if it is an isogeny in the usual sense and* $M_F \to N_F$ *is an isogeny. We say that* M *and* N *are strongly isogenous if there is a virtual* F-*crystal* L *and strong isogenies* $L \to M$ *and* $L \to N$.

<u>Proposition</u> 2.3. i) *A morphism* $M \to N$ *between virtual* F-*crystals is a strong isogeny iff a cone of* $t'(\mathrm{Hodge}(M)) \to t'(\mathrm{Hodge}(N))$ *has cohomology*

of finite length as W-module.

ii) *Strong isogenies are stable by pullbacks and compositions.*

iii) *The relation of being strongly isogenous is an equivalence relation.*

iv) *The Newton and Hodge polygons are the same for strongly isogenous F-crystals.*

v) $T^{i,-i}(t'(\mathrm{Hodge}(M)) = \dim(M_F)^i$.

Proof : By (2.1) to prove i) it is sufficient to show that if $K \in D^b_c(R)$, $\underline{s}(K)$ has finite length cohomology and the cohomology of $K^F_\cdot$ is a finite perfect group scheme then the cohomology of K has finite length. Now $\underline{s}(-)$ and $((-)^F_\cdot)^0$ have Δ-amplitude $[0,1]$ and so we reduce immediately to $K \in \Delta$. The first condition says that K is $\underline{s}$-torsion so we need only show that there is no diagonal domino part. I now claim that for any $L \in \Delta$ $H^1(L^F_\cdot)^0$ is a connected unipotent perfect group scheme and that $L \mapsto H^1(L^F_\cdot)^0$ is exact up to isogeny. The first part is clear by dévissage as $H^1((-)^F_\cdot)^0$ is right exact on Δ. The second part is then clear as dévissage again shows that $H^0(L^F_\cdot)^0$ always is étale. Using this result, dévissage and (III:2.6.1),we see that $T^{i,-i}(L) = \dim H^{-i+1}(L^F_\cdot)^i$ for any L. This shows that $T^{i,-i}(K) = 0$ for all i and hence gives i) and gives also, through (2.1), v).Now i) immediately gives that strong isogenies are stable under compositions as the class of R-complexes with cohomology of finite length as W-module is stable under cones. Let $L \longrightarrow N$ and $M \longrightarrow N$ be two morphisms between F-crystals, suppose that $L \longrightarrow N$ is a strong isogeny and let K be the pullback of $L \longrightarrow N$ and $M \longrightarrow N$. As $\mathrm{Hodge}(-)$ is right adjoint to $(-)^\infty$ it preserves pullbacks and so $\mathrm{Hodge}(K)$ is the pullback of $\mathrm{Hodge}(M) \longrightarrow \mathrm{Hodge}(N)$ and $\mathrm{Hodge}(L) \longrightarrow \mathrm{Hodge}(N)$. This implies that cones of $t'(\mathrm{Hodge}(L)) \longrightarrow t'(\mathrm{Hodge}(N))$ and $t'(\mathrm{Hodge}(K)) \longrightarrow t'(\mathrm{Hodge}(M))$ are isomorphic and so by i) $K \longrightarrow M$ is a strong isogeny. This finishes the proof of ii) and iii) is a formal consequence of ii). Finally the statement concerning Newton polygons is clear and the $m^{i,-i}$

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are the same. By v) the $T^{i,-i}$ are the same and so by definition the $h_W^{i,-i}$ are the same but $h_W^{i,-i} = h^{i,-i}$ as we are dealing with Mazur-Ogus complexes.

If M is an F-crystal then $(M_F)^i$, being a unipotent algebraic group, is isogenous to a sum of $\underline{W}_n$:s and the ordered partition $n_1 \leq \ldots \leq n_r$ of $\dim(M_F)^i$ s.t. $(M_F)^i$ is isogenous to $\bigoplus_{j=1}^{r} \underline{W}_{n_j}$ is a strong isogeny invariant. Hence we get as strong isogeny invariants, besides the ordinary isogeny class of the F-crystal, for each i a sequence $n_{i1} \leq n_{i2} \leq \ldots \leq n_{ir_i}$ of positive integers and we have $T^{i,-i} = \sum_{j=1}^{r_i} n_{ij}$.

<u>Remark</u> : It will be shown elsewhere that these invariants determine the strong isogeny class of a virtual F-crystal.

VI

Examples and Applications

1. As promised in the introduction we will now give some geometric examples. We will begin by studying a special type of inseparable covering. Let k be an algebraically closed field of characteristic $p > 0$ and let X be a smooth and projective surface over k.

<u>Remark</u> : In order to improve on the readibility of the arguments to follow I will often assume stronger hypotheses then necessary for the possibility of an analysis. I leave to the interested reader, if there is one, to weaken conditions.

I will assume given a line bundle L on X and will consider the following two conditions that may be put on L

A) $H^0(X,L^{-i}) = 0$ *for* $1 \leq i \leq 3p-3$ *and* $H^1(X,L^{-i}) = 0$ *for* $1 \leq i \leq 2p-1$

B) $H^0(X,\Omega_X^1 \otimes L^{-i}) = 0$ *for* $1 \leq i \leq p-1$.

i) *Let* α_L *denote the group scheme on* X *which is the kernel of the relative Frobenius morphism* $F : L \to L^p$ *where* L *and* L^p *are considered as additive (smooth) group schemes on* X .

Hence we have an exact sequence of flat group schemes on X

(1.1) $$0 \to \alpha_L \to L \xrightarrow{F} L^p \to 0 .$$

Locally, in the Zariski topology, on X α_L is isomorphic to α_p . We fix once and for all a non-trivial α_L-torsor $\pi : \tilde{X} \to X$ on X . For example, such a torsor is obtained by applying the boundary map obtained from (1.1) to a section $s \in H^0(X,L^p)$ which is not he p'th power of a section

of L. In that case $\pi_*\mathcal{O}_{\tilde{X}} = \bigoplus_{i=0}^{p-1} L^{-i}$ with the obvious multiplication $L^{-i} \otimes L^{-j} \longrightarrow L^{-i-j}$ if $i+j<p$ and $L^{-i} \otimes L^{-j} \longrightarrow L^{-i-j} \xrightarrow{\mathrm{id}\otimes s} L^{-i-j+p}$ if $i+j \geq p$. Locally, in the Zariski topology, the long exact sequence of cohomology obtained from (1.1) shows that any $\tilde{X}$ is of this form. In general it is easy to see that there is a filtration $\mathcal{O}_X = M^0 \subseteq M^1 \subseteq \dots M^{p-1} = \pi_*\mathcal{O}_{\tilde{X}}$ of sub-$\mathcal{O}_X$-modules such that $M^i.M^j \subseteq M^{i+j}$ when $i+j<p$ and $M^i/M^{i-1} = L^{-i}$ $1 \leq i \leq p-1$.

ii) $\tilde{X}$ *is an integral local complete intersection scheme and* $\omega_{\tilde{X}} = \pi^*(\omega_X \otimes L^{p-1})$.

Indeed, to see that $\tilde{X}$ is a local complete intersection it suffices, X being smooth, to show that $\tilde{X} \longrightarrow X$ is a local complete intersection morphism. This is local in the flat topology on X so we may trivialize $\tilde{X}$ so it is enough to prove that $\alpha_L \longrightarrow X$ is an l.c.i morphism. However, α_L is a flat group scheme. In particular depth $\tilde{X} = 2$ so to prove that $\tilde{X}$ is reduced it suffices to prove that it is generically reduced. Consider $\tilde{X}_{red} \longrightarrow \tilde{X} \longrightarrow X$. If X is not generically reduced then as $\tilde{X} \longrightarrow X$ generically is of the form $\mathrm{Spec}(k(X)[t]/(t^p-f)) \longrightarrow \mathrm{Spec}\, k(X)$ we see that $f \in k(X)^p$ and $\tilde{X}_{red} \longrightarrow X$ is birationally an isomorphism. It is finite, as it is the composite of a closed immersion and a finite morphism and X is normal. Therefore $\tilde{X}_{red} \longrightarrow X$ is an isomorphism by Zariski's main theorem. This gives a section $X \xrightarrow{\sim} \tilde{X}_{red} \longrightarrow \tilde{X}$ contrary to the non-triviality of $\tilde{X} \longrightarrow X$. Finally, $\omega_{\tilde{X}} = \pi^*\omega_X \otimes \omega_{\tilde{X}/X}$ so it suffices to show that $\omega_{\tilde{X}/X} = \pi^*(L^{p-1})$. By an easily proven result on group scheme torsors this equals $\pi^*(s^*\omega_{\alpha_L/X})$ where $s : X \longrightarrow \alpha_L$ is the zero section and the adjunction formula applied to the immersion $\alpha_L \longrightarrow L$ gives $s^*\omega_{\alpha_L/X} = L^{p-1}$.

iii) *Suppose* A. *Then* $H^1(X,\mathcal{O}_X) \longrightarrow H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ *is an isomorphism,* $H^2(X,\mathcal{O}_X) \longrightarrow H^2(\tilde{X},\mathcal{O}_X)$ *is a monomorphism and* $\dim_k H^2(\tilde{X},\mathcal{O}_{\tilde{X}})/H^2(X,\mathcal{O}_X) = \sum_{i=1}^{p-1} \dim_k H^2(X,L^{-i})$.

To see this it will suffice to show that $H^0(X,\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)=0=H^1(X,\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$ and that $\dim_k H^2(X,\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)=\sum_{i=1}^{p-1}\dim_k H^2(X,L^{-i})$. Now $\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ has a filtration by the $M^i/\mathcal{O}_X$ with successive quotients L^{-i} for $1\leq i\leq p-1$ and we use condition A.

Let us introduce some notation. If M is an $\hat{R}^0$-module of finite type then there is a largest submodule which is of finite type as W-module. Denote this by TM. As a corollary of iii) we get :

iv) *Suppose* A. *Then* $H^1(X,W\mathcal{O}_X)\longrightarrow H^1(\tilde{X},W\mathcal{O}_{\tilde{X}})$ *is an isomorphism and* $H^2(X,W\mathcal{O}_X)\longrightarrow H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})$ *is a monomorphism whose cokernel is killed by* p. *Furthermore, it induces an isomorphism on* $T(-)$ *and* $T(-)$ *of its cokernel is zero.*

Proof : Let C be a mapping cone of $R\Gamma(X,W\mathcal{O}_X)\longrightarrow R\Gamma(X,W\mathcal{O}_{\tilde{X}})$. Then iii) shows that $H^i(\hat{R}^0/V\hat{R}^0\otimes^L_{\hat{R}^0}C)=0$ if $i\neq 2$. The long exact sequence of the distinguished triangle

$$\longrightarrow C\xrightarrow{V}C\longrightarrow R/VR^0\otimes^L_{\hat{R}}C\longrightarrow$$

and the fact that $H^i(C)$ are separated in the V-topology, being of finite type over $\hat{R}^0$, show that $H^i(C)=0$ if $i\neq 2$ and that V is injective on $H^2(C)$. This shows that $H^1(X,W\mathcal{O}_X)\longrightarrow H^1(\tilde{X},W\mathcal{O}_{\tilde{X}})$ is an isomorphism and that $H^2(X,W\mathcal{O}_X)\longrightarrow H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})$ is injective. We now want to show that its cokernel, that is $H^2(C)$, is killed by p. As $p=VF$ it will be sufficient to show that F maps $H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})$ into $H^2(X,W\mathcal{O}_X)$. It is clear that the Frobenius $F:X\longrightarrow X$ trivializes $\tilde{X}$ so that there exists a mapping $X\longrightarrow\tilde{X}$ such that the composite $X\longrightarrow\tilde{X}\xrightarrow{\pi}X$ equals F. It is easy to see that $\tilde{X}\xrightarrow{\pi}X\longrightarrow\tilde{X}$ is also $F:\tilde{X}\longrightarrow\tilde{X}$. Now $F:H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})\longrightarrow H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})$ is induced by functoriality from $F:\tilde{X}\longrightarrow\tilde{X}$ so it factors through $\pi:H^2(X,W\mathcal{O}_X)\longrightarrow H^2(\tilde{X},W\mathcal{O}_{\tilde{X}})$ which is what we want. Hence $H^2(C)$ is killed

by p with V injective and so contains no submodule of finite type over W. Therefore $TH^2(\widetilde{X},WO_{\widetilde{X}})$ is contained in $H^2(X,WO_X)$ and thus equals $TH^2(X,WO_X)$.

v) *Assume* A. *Then* $\dim_k(H^2(\widetilde{X},WO_{\widetilde{X}})/H^2(X,WO_X))/V(H^2(\widetilde{X},W_{\widetilde{X}})/H^2(X,W_X)) = \sum_{i=1}^{p-1} \dim_k H^2(X,L^{-i})$.

This follows from iii) and iv).

Let us consider $H^1(\widetilde{X},\mathbb{G}_m)$. Put $N := \pi_*\mathbb{G}_m/\mathbb{G}_m$. If we pull back $\widetilde{X}$ by π then it becomes trivial dans $\pi^*\widetilde{X} = \mathrm{Spec}(\bigoplus_{i=0}^{p-1} L^{-i})$ with multiplication the obvious one ; $L^{-i}\otimes L^{-j} \to L^{i-j}$ if $i+j<p$ and zero if not. Also $\pi^*N = 1+\bigoplus_{i=1}^{p-1} L^{-i}$ as multiplicative group is through exp and log isomorphic to $\bigoplus_{i=1}^{p-1} L^{-i}$.

vi) *Assume* A. $H^1(X,\mathbb{G}_m) \to H^1(\widetilde{X},\mathbb{G}_m)$ *is an isomorphism and* $H^2(X,\mathbb{G}_m) \to H^2(\widetilde{X},\mathbb{G}_m)$ *is injective with cokernel killed by* p.

Indeed, it is clear that we need to prove that $H^0(X,N) = H^1(X,N) = 0$ and that $H^2(X,N)$ is killed by p. As π is faithfully flat and π^*N, as we have just seen, is killed by p the last statement follows. Again as π is faithfully flat we get a Čech spectral sequence

$$H^i(\widetilde{X}^{(j)},N) \Longrightarrow H^{i+j-1}(X,N)$$

where $\widetilde{X}^{(j)} := \widetilde{X}\times_X \widetilde{X}\times_X\cdots\times_X\widetilde{X}$ (j times)

so it will be sufficient to show that

$$H^0(\widetilde{X},\pi^*N) = H^1(\widetilde{X},\pi^*N) = H^0(\widetilde{X}^{(1)},\pi_2^*N) = 0$$

where $\pi_2 : \widetilde{X}^{(2)} \to X$. Now $\pi^*N \simeq \bigoplus_{i=1}^{p-1} L^{-i}$, $\widetilde{X}^{(2)} = \alpha_L\times_X\widetilde{X}$ with π_2 the composite of the projection q_2 onto the second factor and π. Hence we need to prove that $H^0(\widetilde{X},\pi^*L^{-i}) = H^1(\widetilde{X},\pi^*L^{-i}) = H^0(\alpha_L\times_X\widetilde{X}, q_2^*\pi^*L^{-i}) = 0$ for $1\le i\le p-1$. Now $\pi_*\pi^*L^{-i} = L^{-i}\otimes_{O_X}\pi_*O_{\widetilde{X}}$ and $\pi_*q_{2*}q_2^*\pi^*L^{-i} = \bigoplus_{j=1}^{p-1} L^{-i}\otimes L^{-j}\otimes\pi_*O_{\widetilde{X}}$ so

using $H^0(\tilde{X},\pi^*L^{-i}) = H^0(X,\pi_*\pi^*L^{-i})$ etc., the filtration M^i of $\pi^*\mathcal{O}_{\tilde{X}}$ and condition A we are through.

We get immediately the following consequence :

vii) *Assume* A . *Then* $H^1(X,\mathbb{Z}_p(1)) \longrightarrow H^1(\tilde{X},\mathbb{Z}_p(1))$ *and* $H^2(X,\mathbb{Z}_p(1)) \longrightarrow H^2(\tilde{X},\mathbb{Z}_p(1))$ *are isomorphisms.*

<u>Proof</u> : Recall that $H^i(Y,\mathbb{Z}_p(1)) := \varprojlim_n H^i(Y,\mu_{p^n})$. The exact sequences $0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 0$ give short exact sequences

$$0 \longrightarrow \widehat{H^{i-1}(Y,\mathbb{G}_m)} \longrightarrow H^i(Y,\mathbb{Z}_p(1)) \longrightarrow T_p(H^i(Y,\mathbb{G}_m)) \longrightarrow 0$$

where $\hat{M} := \varprojlim_n M/p^nM$ and $T_pM := \varprojlim_n\{{}_{p^n}M,p\}$. Hence vii) follows from v and the 5-lemma once we have shown that $T_p(H^2(X,\mathbb{G}_m)) \longrightarrow T_p(H^2(\tilde{X},\mathbb{G}_m))$ is an isomorphism. However, $T_p(-)$ is left exact and zero on a group killed by p so this again follows from vi).

We can associate to $\tilde{X}$ a mapping $\varphi : L^{-p} \longrightarrow \Omega^1_X$. When L is trivialized and $\tilde{X}$ is the boundary of $s \in L^p$ corresponding to $s' \in \mathcal{O}_X$ under the trivialization then $t \in L^{-p}$ is taken to $t'ds'$ where $t' \in \mathcal{O}_X$ corresponds to t . It is easy to see that the set of zeroes of $L^{-p} \longrightarrow \Omega^1_X$ equals the projection to X of the singular set of $\tilde{X}$. As $\tilde{X}$ has depth 2 the zeroes of φ are isolated iff $\tilde{X}$ is normal.

Let us introduce hypothesis C : *the zeroes of* φ *are isolated and simple.*

It is clear that this is equivalent to the condition that locally, in the étale topology, the s' above has the form xy where x,y is a coordinate system on X . Hence the singularities of $\tilde{X}$ have the form $Z^p = xy$ and so are rational double points of type A_{p-1} .

viii) *Assume* A *and* C . *The number of singular points on* $\tilde{X}$ *is* $(L.L)p^2 + (\omega_X.L)p + c_2(X)$.

This is simply the formula for the number of zeroes of $\varphi : L^{-p} \longrightarrow \Omega^1_X$ when all the zeroes are isolated.

Let $\rho : X' \longrightarrow \widetilde{X}$ be a minimal resolution of the singularities of $\widetilde{X}$.

ix) *Assume* A *and* C.

a) $\omega_{X'} = \rho^*\pi^*(\omega_X \otimes L^{p-1})$.

b) $H^i(\widetilde{X}, WO_{\widetilde{X}}) \longrightarrow H^i(X', WO_{X'})$ *is an isomorphism for all* i.

c) $H^1(\widetilde{X}, \mathbb{Z}_p(1)) \longrightarrow H^1(X', \mathbb{Z}_p(1))$ *is an isomorphism and* $H^2(\widetilde{X}, \mathbb{Z}_p(1)) \longrightarrow H^2(X', \mathbb{Z}_p(1))$ *is injective with cokernel a free* $\mathbb{Z}_p$-*module of rank equal to* $(p-1)((L.L)p^2 + (\omega_X.L)p + c_2(X))$.

Indeed, as the singularities of $\widetilde{X}$ are rational $R\rho_* O_{X'} = O_{\widetilde{X}}$ which gives $R\rho_* WO_{X'} = WO_{\widetilde{X}}$ which in turn gives b). As $\widetilde{X}$ has in fact only rational double points $\rho^*(\omega_{\widetilde{X}}) = \omega_{X'}$ so we get a). To prove c) we consider the spectral sequence

$$H^i(\widetilde{X}, R^j\rho_*\mu_{p^n}) \Longrightarrow H^{i+j}(X', \mu_{p^n}) .$$

I claim first that $R^1\rho_*\mu_{p^n} = 0$. To prove this we are reduced to proving that multiplication by p^n is injective on $\mathrm{Pic}\, X'_S$ where S is the spectrum of a local ring and $S \longrightarrow \widetilde{X}$ a flat morphism. As the singularities of $\widetilde{X}$ are rational $H^1(X'_S, O_{X'_S}) = 0$ so this follows from [Li: Thm 12.1]. This immediately gives that $H^1(\widetilde{X}, \mathbb{Z}_p(1)) \longrightarrow H^1(X', \mathbb{Z}_p(1))$ is an isomorphism and that $\tau : H^2(\widetilde{X}, \mathbb{Z}_p(1)) \longrightarrow H^2(X', \mathbb{Z}_p(1))$ is injective. As $H^1(\widetilde{X}, \mu_p) \longrightarrow H^1(X', \mu_p)$ is an isomorphism and $H^2(\widetilde{X}, \mu_p) \longrightarrow H^2(X', \mu_p)$ an injection, τ is surjective on the kernel of p and injective on the cokernel which shows that $\mathrm{Coker}\,\tau$ is torsion free. I claim that $T_pH^2(\widetilde{X}, \mathbb{G}_m) \longrightarrow T_pH^2(X', \mathbb{G}_m)$ is an isogeny. Indeed by vii) we can replace $\widetilde{X}$ by X and then we have a diagram

$$\begin{array}{ccccc} \bar{\bar{X}} & \longrightarrow & \bar{X} & \longrightarrow & X \\ \searrow & & \searrow & & \downarrow F \\ & \widetilde{X} & \xrightarrow{F} & \widetilde{X} & \longrightarrow X \end{array}$$

where $\bar{X} \longrightarrow X$ and $\bar{\bar{X}} \longrightarrow \tilde{X}$ are blowing ups. As $T_p H^2(-,\mathbb{G}_m)$ is a birational invariant on smooth, proper surfaces we conclude. Hence the rank of Coker τ equals the rank of $\widehat{\mathrm{Pic}}\, X'$ minus the rank of $\widehat{\mathrm{Pic}}\, \tilde{X}$ and this difference is equal to the number of irreducible components of the exceptional fibers. As we have an A_{p-1}-singularity we get $p-1$ such for each singular point so we conclude by viii).

We can now compare invariants of X and X'.

x) *Assume* A *and* C .
a) $c_1^2(X') = p(c_1^2(X) + 2(p-1)(\omega_X . L) + (p-1)^2(L.L))$.
b) $H^1(X,W\mathcal{O}_X) \longrightarrow H^1(X',W\mathcal{O}_{X'})$ *is an isomorphism and* $H^2(X,W\mathcal{O}_X) \longrightarrow H^2(X',W\mathcal{O}_{X'})$ *is an injection with a* V*-torsion free cokernel killed by* p . *The dimension of the cokernel as Dieudonné module is* $\frac{p(p-1)(p-2)}{6}(L.L) + \frac{p(p-1)}{4}(\omega_X \otimes L^{-1}.L) + (p-1)\chi$.
c) $H^1(X,\mathbb{Z}_p(1)) \longrightarrow H^1(X',\mathbb{Z}_p(1))$ *is an isomorphism and* $H^2(X,\mathbb{Z}_p(1)) \longrightarrow H^2(X',\mathbb{Z}_p(1))$ *is an injection with torsion free cokernel of rank* $(p-1)((L.L)p^2 + p(\omega_X.L) + c_2(X))$.

This is just putting together ix), v), vii), R-R and A to compute $\chi(L^{-i})$.

<u>Remark</u> : c) was essentially proven by W. Lang under the stronger assumption B (cf. [La]).

xi) *Assume* A *and* B . *Then* $H^0(X,\Omega_X^1) = H^0(\tilde{X},\Omega_{\tilde{X}}^1)$.

Indeed, we have an exact sequence on $\tilde{X}$

$$0 \longrightarrow \pi^*L^{-p} \longrightarrow \pi^*\Omega_X^1 \longrightarrow \Omega_{\tilde{X}}^1 \longrightarrow \pi^*L^{-1} \longrightarrow 0 .$$

It is clear that we are finished if $H^0(\pi^*L^{-p}) = H^0(\pi^*L^{-1}) = H^1(\pi^*L^{-p}) = 0$ and $H^0(\Omega_{\tilde{X}}^1) = H^0(\pi^*\Omega_X^1)$. By projecting down to X we see that the first part is implied by A and as $\pi_*\pi^*\Omega_X^1 = \pi_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \Omega_X^1$ the second is implied by B if

we use the filtration $\{M^i\}$.

xii) *Assume* C . *Then every closed form in* $H^0(\Omega^1_{X'})$ *is in the image of* $H^0(\Omega^1_{\tilde{X}})$.

Proof : The statement is Zariski local on X so we may assume that $\tilde{X}$ is of the form $\mathrm{Spec}(\mathcal{O}_X[t]/(t^p-f))$ for $f \in \mathcal{O}_X$. Then we have a cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbb{A}^1 & \xrightarrow{F} & \mathbb{A}^1 \end{array}$$

where F is the Frobenius.

We have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \rho_*\Omega^1_{\mathbb{A}^1} & \longrightarrow & \rho_*\Omega^1_{X'} & \longrightarrow & \rho_*\Omega^1_{X'/\mathbb{A}^1} & \longrightarrow & 0 \\ & & s\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega^1_{\mathbb{A}^1} & \longrightarrow & \Omega^1_{\tilde{X}} & \longrightarrow & \Omega^1_{\tilde{X}/\mathbb{A}^1} & \longrightarrow & 0 \end{array}$$

and it is sufficient to show that the image in $\rho_*\Omega^1_{X'/\mathbb{A}^1}$ of a closed form cones from $\Omega^1_{\tilde{X}/\mathbb{A}^1}$. However, [Sz:Exp. V Lemme 2] shows that it actually comes from $\Omega^1_{X/\mathbb{A}^1}$ as the global assumptions are never used there.

xiii) *Assume* A , B *and* C .

Then the closed 1-*forms of* X' *are exactly those in the image of the injective mapping* $H^0(X,\Omega^1_X) \longrightarrow H^0(X',\Omega^1_{X'})$.

As it is obvious that all forms in the image are closed this follows from xi) and xii).

We will now further specialize to $X = \mathbb{P}^2$ and $L = \mathcal{O}(n)$ $n > 0$. Then A and B are fulfilled and a simple dimension count shows that we can find $s \in \mathcal{O}(pn)$ such that $\mathcal{O}(-pn) \longrightarrow \Omega^1_{\mathbb{P}^2}$ has only isolated simple singularities.

From x) we then get that $H^1(X',W\mathcal{O}_{X'}) = H^0(W\Omega^1_{X'}) = H^0(W\Omega^2_{X'}) = 0$, that $H^2(W\mathcal{O}_{X'})$ is killed by p and is without finite torsion, that $H^1(W\Omega^1_{X'})$ is torsion free of slope 0 and that it equals $\operatorname{Pic} X' \otimes_{\mathbb{Z}} W$. Now $\operatorname{Pic} X'$ contains the inverse image of a hyperplane and all the exceptional divisors. The square of the inverse image of a hyperplane is p and as the singularities are of type A_{p-1} each connected component of the exceptional divisors gives an intersection matrix of determinant $(-1)^{p-1}p$. Hence the p-adic ordinal of the discriminant of the group of cycles generated by the cycles that has just been described is $1 + p^2n^2 - 3pn + 3$. By ix) this group has finite index in $\operatorname{Pic} X'$ so $\operatorname{Pic} X'$ has discriminant of p-adic ordinal $\leq 1+p^2n^2-3pn+3$ and as $H^1(W\Omega^1_{X'}) = \operatorname{Pic}(X') \otimes W$ so has $H^1(X,W\Omega^1)$. If $M := (H^2(W\mathcal{O}_{X'}) \xrightarrow{d} H^2(W\Omega^1_{X'}))$, which is a domino, has type σ then by (III:3.3;4.6.1) $2\sum_{i\in\mathbb{Z}} i\sigma(i) \leq 1+p^2n^2-3pn+3$. Now, as $R\Gamma(X',\Omega^\cdot_{X'}) = R_1 \otimes^L_R R\Gamma(X',W\Omega^\cdot_{X'})$ (cf. [Il-Ra]) we have $H^2(X',\Omega^1) = H^1(M,dV)$ and so $h^0(\Omega^1_{X'}) = h^2(\Omega^1_{X'}) = \sum_{i\in\mathbb{N}} \sigma(-i)(i+1)$ by (III:3.3). As $\sum_{i\in\mathbb{Z}} \sigma(i)(-i+1) =$

$\frac{p(p-1)(p-2)}{6}n^2 - \frac{p(p-1)}{4}(n+3) + p-1 - \sum_{i\in\mathbb{Z}} i\sigma(i)$ by (x),b) we get :

xiv) $X = \mathbb{P}^2$, $L = \mathcal{O}(n)$, $n > 0$. $h^0(\Omega^1_{X'}) = \sum_{i\in\mathbb{N}} \sigma(-i)(i+1) \geq \sum_{i\in\mathbb{Z}} \sigma(i)(-i+1) \geq$

$\frac{p(p-1)(p-2)}{6}n^2 - \frac{p(p-1)}{4}(n+3) + p-1 - \frac{1}{2}(p^2n^2+3pn-4) = \frac{p[(p-1)(p-2)-3p]}{6}n^2 - \frac{p(p+5)}{4}n - \frac{3p^2-7p}{4} + 1$.

Also by xi) *and* xii) $d : H^0(\Omega^1_{X'}) \to H^0(\Omega^2_{X'})$ *is injective and so by duality* $d : H^2(\mathcal{O}_{X'}) \to H^2(\Omega^1_{X'})$ *is surjective. This shows by* (III:3.3) *that* $\sigma(i) = 0$ *if* $i < 0$ *and in particular that there is no exotic torsion so there is no crystalline torsion.*

2. We will now go on to discuss the phenomenon of exotic torsion. Let us first note that we can find examples of a smooth and proper scheme X over $\operatorname{Spec} W$ such that its special fiber has exotic torsion. Indeed, we may take

as example the product of an Igusa surface with itself which will have exotic torsion in degree 3 (cf. [Ek 2 : III 8:10]). If we confine our attention to degree 2 the situation becomes more intricate. It is clear by Kummer theory that the p-torsion of $H^2(X',\mathbb{Z}_p)$, where X' is a geometric generic fiber of X, equals the p-torsion of $\mathrm{Pic}^{\tau}X'/\mathrm{Pic}^{0}X'$. By [Ra] there is an abelian subscheme $A \subseteq \mathrm{Pic}^{\tau}X$ such that $N = \mathrm{Pic}^{\tau}X/A$ is finite and flat over W. *Hence the generic fiber of* N *equals* $\mathrm{Pic}^{\tau}X'/\mathrm{Pic}^{0}X'$.

I claim that the Dieudonné module of the special fiber $\bar{N}$ *equals the finite torsion of* $H^2(\bar{X}/W)$ *where* $\bar{X}$ *is the special fiber of* X.

Indeed, by (I:2.2.2) this finite torsion is the sum of the V-torsion of $H^2(\bar{X},W\mathcal{O}_{\bar{X}})$ and the p-torsion of $H^1(\bar{X},W\Omega^1_{\bar{X}})$. The first part is the Dieudonné module of the connected part of $\bar{N}$ (cf. [Il1:II]) and the second part the Dieudonné module of the étale part of $\bar{N}$ (cf. loc. cit.).

Hence we see that the length of the p-torsion of $H^2(X',\mathbb{Z}_p)$ equals the length of the finite torsion of $H^2(\bar{X}/W)$ and so we see that the length of the p-torsion of $H^2(X',\mathbb{Z}_p)$ equals the length of the torsion of $H^2(\bar{X}/W)$ iff $\bar{X}$ has no exotic torsion in degree 2 thus relating the existence of exotic torsion to an old problem of Grothendieck (cf. [Il 2]).

On the contrary if we lift with much ramification $(e \geq p-1)$ then we can get more crystalline torsion than classical torsion for two reasons. First the example (cf. [Ra]) of Raynaud gives a non-flat N and so the order of N at the generic point is strictly smaller than its order at the special point and so the length of $H^2(X',\mathbb{Z}_p)$ is strictly smaller than the length of the finite torsion of $H^2(\bar{X}/W)$. Secondly we will see in a moment that we may also get exotic torsion.

<u>Remark</u> : Raynaud's example is constructed starting from a finite $\mathbb{F}_p$-vector space V, a non-degenerate alternating pairing φ on V and the associated Heisenberg group N. If one lets $\dim V \geq 6$ and considers the split extension $G = N \ltimes \mathrm{Sp}(\varphi)$ then we may replace N by G in Raynaud's argument and

then the general fiber of X will have a perfect fundamental group so $H^2(X',\mathbb{Z}_\ell)$ will be torsion free for *all* primes ℓ (including p) but $H^2(\bar{X}/W)$ will have torsion.

As our first example of how to produce exotic torsion let us again consider a line bundle L on a smooth and proper surface fulfilling condition A of 1 and $\pi : X' \longrightarrow X$ the resolution of an α_L-torsor for which $L^{-p} \longrightarrow \Omega^1_X$ has only simple isolated zeroes.

I claim first that π^* *takes* $H^0(X,Z_n\Omega^1_X)$ *into* $H^0(X',Z_{n+1}\Omega^1_{X'})$.

Indeed, this only uses the fact that π is inseparable ; $Z_{n+1}\Omega^1$ is the kernel of $dC : Z_n\Omega^1 \longrightarrow \Omega^2$ and $\pi^* : H^0(X,\Omega^2_X) \longrightarrow H^0(X',\Omega^2_{X'})$ is zero as π is inseparable. *I further claim that* $\pi^* : H^0(X,Z_\infty\Omega^1_X) \longrightarrow H^0(X',Z_\infty\Omega^1_{X'})$ *is an isomorphism.* By [Il 1:0.2.5.3.3-5] it will suffice to show that π^* induces an isomorphism on global sections of $\mathbb{G}_m/p\mathbb{G}_m$ and $B_n\Omega^1$ for all n. The first part follows from (1;x). For the second part we get from [Il 1: I.3.11.4] that it will be sufficient to show that π induces an isomorphism on the kernel of F on $H^1(-,W_n\mathcal{O})$. However, from (1;x) it follows that π^* induces an isomorphism already on $H^1(-,W_n\mathcal{O})$. It is clear from the proof of xi) that A implies that $\pi^* : H^0(X,\Omega^1_X) \longrightarrow H^0(X',\Omega^1_{X'})$ is injective.

We therefore see that $\dim_k H^0(X',Z_{n+1}\Omega^1_{X'})/H^0(X',Z_\infty\Omega^1_{X'}) \geq \dim_k H^0(X,Z_n\Omega^1_X)/H^0(X,Z_\infty\Omega^1_X)$.

For $n=0$ *we get that if* X *has non-closed* 1*-forms then, by* [Il 1: II.6.16], X' *has exotic torsion.*

We can take as X *one of the coverings of* $\mathbb{P}^2$ *constructed at the end of* 1 *, from the formula given there we see that if* $p \geq 7$ *then we may always find such an* X *with non-closed* 1*-forms.*

As another type of example let us consider a smooth and proper surface X with a line bundle L s.t $H^1(X,L^{-1}) = 0$ and s.t there is a Lefchetz pencil

$V \subseteq H^0(X,L)$. Let $\varphi : Y \longrightarrow \mathbb{P}(V) = \mathbb{P}^1$ be this pencil (where Y is a blowing up of X). Then $R^1\varphi_* \mathcal{O}_Y$ is a locally free sheaf on $\mathbb{P}^1$ hence isomorphic to $\bigoplus_i \mathcal{O}(r_i)$ for some integers r_i. *I claim that* $r_i \leq 0$ *for all* i.

Indeed, let $C \subseteq X$ be a curve in $|V|$. Then the strict transform of C on X' is isomorphic to C as only smooth points of C are blown up and we have a commutative diagram

$$\begin{tikzcd} H^1(X,\mathcal{O}_X) \arrow[d, "\wr"] \arrow[dr] & \\ H^1(Y,\mathcal{O}_Y) \arrow[r] & H^1(C,\mathcal{O}_C) \end{tikzcd}$$

By assumption $H^1(X,\mathcal{O}_X) \longrightarrow H^1(C,\mathcal{O}_C)$ is injective and hence so is $H^1(Y,\mathcal{O}_Y) \longrightarrow H^1(C,\mathcal{O}_C)$. Let $x := \varphi(C)$. Then $H^1(Y,\mathcal{O}_Y) \longrightarrow H^1(C,\mathcal{O}_C)$ is (canonically) isomorphic to $H^0(\mathbb{P}^1, R^1\varphi_*\mathcal{O}_Y) \longrightarrow (R^1\varphi_*\mathcal{O}_Y)_x$ which therefore is injective. This clearly implies that $r_i \leq 0$ for all i. Let now $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a non-constant morphism. We then get a diagram

$$\begin{tikzcd} Y'' \arrow[dr, "\tau"] & & \\ & Y' \arrow[r] \arrow[d, "\rho"] & Y' \arrow[d, "\varphi"] \\ & \mathbb{P}^1 \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

where the square is cartesian and τ a (minimal) resolution of singularities. As $R^1\varphi_*\mathcal{O}_Y \simeq \bigoplus_i \mathcal{O}(r_i)$, $r_i \leq 0$, it is clear that $H^0(\mathbb{P}^1, R^1\varphi_*\mathcal{O}_Y) \longrightarrow H^0(\mathbb{P}^1, f^*R^1\varphi_*\mathcal{O}_Y)$ is an isomorphism and irrespective of the values of the r_i $H^1(\mathbb{P}^1, R^1\varphi_*\mathcal{O}_Y) \longrightarrow H^1(\mathbb{P}^1, f^*R^1\varphi_*\mathcal{O}_Y)$ is a monomorphism. By base change $f^*R^1\varphi_*\mathcal{O}_Y = R^1\rho_*\mathcal{O}_Y$ and as is well known $R\tau_*\mathcal{O}_{Y''} = \mathcal{O}_{Y'}$. *Hence we get that* $H^1(Y,\mathcal{O}_Y) \longrightarrow H^1(Y'',\mathcal{O}_{Y''})$ *is an isomorphism and* $H^2(Y,\mathcal{O}_Y) \longrightarrow H^2(Y'',\mathcal{O}_{Y''})$ *is a monomorphism.*

As in the proof of (1,iv) this implies that $H^1(Y, W_n\mathcal{O}) \longrightarrow H^1(Y'', W_n\mathcal{O})$ is an isomorphism for all n and so as above that $H^0(Y, B_\infty\Omega^1_Y) \longrightarrow H^0(Y'', B_\infty\Omega^1_{Y''})$ is an isomorphism. *I claim that there is a bound for the dimension of* $H^0(Y'', Z_\infty\Omega^1_{Y''})$ *which depends only on* φ *not on* f.

Indeed, we have taken care of $B_\infty\Omega^1$ so as above it suffices to bound the order of ${}_p\mathrm{Pic}\, X^{(3)}$. It is clear that this equals ${}_p\Gamma(k(\mathbb{P}^1), \mathrm{Jac}(Y''/\mathbb{P}^1))$ whose order clearly is bounded by the order of the kernel of p on the Jacobian of a geometric generic fiber of φ so in particular bounded by something depending only on φ.

Let now f be F^i and let $X^{(p^i)}$ be the resolution of the pullback of Y by F^i. Then as we have seen the dimensions of the $H^0(X^{(p^i)}, Z_\infty\Omega^1)$ are bounded.

If we can show: (*) For every n *the dimensions of* $H^0(X^{(p^i)}, Z_n\Omega^1) \longrightarrow \infty$ *we get a situation similar to the one discussed, on p. 111.*

By the same argument as before to show (*) it suffices to show that the $H^0(X^{(p^i)}, \Omega^1) \longrightarrow \infty$ but this is proved in [Sz:Exp. 5].

<u>Remark</u> : By choosing X and V suitably we may make all the $X^{(p^i)}$ liftable. Indeed if we start with X and V lifted the Lefschetz pencil $\varphi : X' \longrightarrow \mathbb{P}^1$ lifts and as F^i on $\mathbb{P}^1$ lifts the pullback by F^i of φ lifts. The singularities of this pullback are rational so we may simultaneously resolve the singularities in the fibers of the lifting (cf. [Ar]) so the $X^{(p^i)}$ lift. In the last step we may be forced to ramify the base of the lifting so even if we start with an unramified lifting, which we certainly may, we have no guarantee that we will end up with one.

3. Let now X be a smooth and proper variety over a perfect field k of characteristic $p > 0$. We will say that X fulfills RH if there is a smooth and proper family $T \longrightarrow S$ with S of finite type over $\mathbb{F}_p$, a morphism $\varphi : \operatorname{Spec} \bar{k} \longrightarrow S$ such that $X_{\bar{k}}$ is isomorphic to the pullback along φ of T and such that for every closed point x of S the fiber T_x has the property that the eigenvalues of the Frobenius morphism F_x with respect to $k(x)$ on $H^i_{cris}(T_x/W(k(x)))$ are algebraic with all its absolute values equal to $|k(x)|^{i/2}$ for all i. It is no doubt true that X always fulfills RH but at the moment this is known only when X is dominated by a smooth and projective variety (possibly over an extension of k) (cf. [Ka-Me]) and so in particular when $\dim X \leq 3$ by resolution of singularities.

Lemma 3.1. i) *Let* X *be of pure dimension* N. *Then* $m^{i,j}(X) = m^{N-i,N-j}(X)$; $T^{i,j}(X) = T^{N-i-2,N-j+2}(X)$ *for all* i,j.

ii) *If* X *satisfies* RH *then* $m^{i,j}(X) = m^{j,i}(X)$ *for all* i,j.

Indeed, the first part of i) comes from Poincaré duality for crystalline cohomology (cf. [Be 1:VII.2.1.3]) and the second part is [Ek 1:IV:Cor. 3.5.1] (it also follows from [Be 2]). As for ii) by constructibility for crystalline cohomology we reduce to the case where k is a finite field. Let $\alpha_1, \alpha_2, \ldots \alpha_r$ be the eigenvalues with multiplicities of the Frobenius wrt k on $H^i(X/W(k))$ and let $v(-)$ be a p-adic valuation on $W(k)[\alpha_1, \ldots \alpha_r]$ normalized such that $v(|k|) = 1$. It will then suffice to show that for every $\lambda \in \mathbb{Q}$ the number of α_s with $v(\alpha_s) = \lambda$ equals the number of α_s with $v(\alpha_s) = i - \lambda$. Indeed, it follows from the hypotheses that $\prod_s (t - \alpha_s) \in \mathbb{Z}[t]$ and that $\alpha_s \bar{\alpha}_s = q^i$ where $\bar{\alpha}_s$ is a complex conjugate. Hence $\alpha_s \longrightarrow q^i/\alpha_s$ is a permutation of $\{\alpha_1, \ldots \alpha_r\}$ which gives the result.

Remark : As $b_n = \sum_{i+j=n} m^{i,j}$ it follows that, when X fulfills RH, b_n is even when n is odd. As in that case $b_n = \operatorname{rk}_{\mathbb{Z}_\ell} H^n(X_{\bar{k}}, \mathbb{Z}_\ell)$ this answers a

question by Deligne (cf. [De:IV.4.1.4]) when X fulfills RH and so probably always.

Proposition 3.2. i) *If* X *has pure dimension* N *then* $h_W^{i,j}(X) = h_W^{N-i,N-j}(X)$.
ii) *If* X *fulfills* RH *then* $h_W^{i,j}(X) = h_W^{j,i}(X)$ *for all* i,j *with* $i+j=n$ *iff* $T^{i,j} = T^{j-2,i+2}$ *for all* i,j *with* $i+j=n$.

It is clear that this follows from 3.1 and the definition of $h_W^{i,j}$.

Corollary 3.3. i) $h_W^{i,j} = h_W^{j,i}$ *if* $i+j \leq 1$.
ii) *If* X *fulfills* RH *then* $h_W^{i,j} = h_W^{j,i}$ *for* $i+j \leq 2$.
iii) *If* $\dim X \leq 3$ *then* $h_W^{i,j} = h_W^{j,i}$ *for all* i,j .

Proof : If $i+j=0$ then clearly $h_W^{i,j} = h_W^{j,i}$. If $i+j=1$ then the only non-trivial equality if $h_W^{1,0} = h_W^{0,1}$. However $h_W^{0,1} = m^{0,1} = \dim_k H^1(X,WO)/V$ which is the dimension of the formal group associated to the abelian variety $(\mathrm{Pic}^0 X)_{red}$ and so equal to the dimension of $(\mathrm{Pic}^0 X)_{red}$ which is $b_1/2$. As $h_W^{0,1} + h_W^{1,0} = b_1$ we get $h_W^{0,1} = b_1/2 = h_W^{1,0}$. As for ii) we are left with $i+j=2$ by i) and so by (3.2) we have to verify that $T^{i,j} = T^{j-2,i+1}$ when $i+j=2$. In this case $T^{i,j} = 0$ unless $(i,j) = (0,2)$ and $T^{0,2} = T^{2-2,0+2}$ so there is nothing to prove. For iii) finally we have already noted that X fulfills RH and so it remains to prove $T^{i,j} = T^{j-2,i+2}$. By ii) and (3.2 i) we may also assume that X is purely 3-dimensional. Then the only non-trivial inequality is $T^{0,3} = T^{1,2}$ but this follows from (3.1 i).

Remark : I don't see any reason why $h_W^{i,j} = h_W^{j,i}$ should be true in general.

If now $\dim X \leq 1$ then X is Mazur-Ogus so $h_W^{i,j} = h^{i,j}$. If $\dim X = 2$ and X is geometrically connected then (3.2 i), (3.3 iii), (IV:3.2) and Noether's formula give

$$h_W^{0,0} = h_W^{2,2} = 1$$

$$h_W^{0,1} = h_W^{1,0} = h_W^{1,2} = h_W^{2,1} = b_1/2$$

$$h_W^{0,2} = h_W^{2,0} = c_2/12 + c_1^2/12 + b_1/2 - 1$$

$$h_W^{1,1} = b_1 + 5c_2/6 - c_1^2/6 \ .$$

From this we can draw several conclusions. First we see that the $h_W^{i,j}$ are deformation invariants and that they are the same as the Hodge numbers for a lifting to characteristic zero (as those Hodge numbers can be expressed in the same way in terms of the Chern and Betti numbers). This is no longer true in higher dimensions. If we let $p=2$ and E_1 and E_2 be an ordinary resp. supersingular elliptic curve, $I = E_1 \times E_2/\langle\sigma\rangle$ where σ acts by $\sigma(x,y) = (x+\alpha,-y)$ where $\alpha \in E_1(k)$ is of order 2 (I is "the" Igusa surface) and we consider $I \times I$ then it is shown in [Ek 2] that $h_W^{0,2}(I\times I) = 2$. If we lift E_1 and E_2 and construct in the same way I' in characteristic 0 then $h^{0,2}(I'\times I') = 1$. Similarly, if we deform E_2 to an ordinary curve then $h_W^{0,2}(I'\times I') = 2$ for the deformation. By taking a suitable hyperplane section we get a 3-dimensional example.

Furthermore, if X, which again is a surface, can be lifted, or is Hodge-Witt or is Mazur-Ogus then $h_W^{1,1} \geq 0$ as it equals respectively $h^{1,1}$ (lifting), $m^{1,1}$ or $h^{1,1}$. However, as observed by Szpiro (cf. [Sz]) if $X \longrightarrow C$ is a non-isotrivial smooth fibration with C a smooth curve then for the sequence $X^{(n)} := X \times_C C$, with $F^n : C \longrightarrow C$ the second projection, of smooth and proper surfaces, the Betti numbers are constant and $c_1^2 \longrightarrow \infty$ so $h_W^{1,1} \longrightarrow -\infty$ and we have examples of surfaces which can be deformed to neither liftable nor Hodge-Witt nor Mazur-Ogus surfaces.

Corollary 3.3 enables us to give a proof of a result of Nygaard which very closely resembles the proof given by Serre in characteristic 0.

Proposition 3.4.[Ny 1] *Let* X *be a smooth and proper* 3-*fold and suppose that there exists a dominant rational separable mapping* $\varphi : \mathbb{P}^3 \dashrightarrow X$. *Then* $\chi(\mathcal{O}_X) = 1$.

Proof : The existence of φ implies that $h^{i,0}(X) = 0$ for $i > 0$. Hence by (IV:3.3) for $i > 0$ $h_W^{i,0} = 0$ as $h_W^{i,0} \geq 0$ and by (3.3) $h_W^{0,i} = 0$ for $i > 0$. Then Crew's formula gives $\chi(\mathcal{O}_X) = \sum_i (-1)^i h_W^{0,i} = 1$.

Remark : Only a very small part of the general theory is really needed in the proof.

4. We will end by giving a generalization of a part of Nygaard's proof of the Rudakov-Shafarevich theorem.

Theorem 4.1. *Let* X *be a smooth and proper surface over* k , *perfect of characteristic* $p > 0$ *and suppose that the rank of the Néron-Severi group of* X *equals* $b_2(X)$. *If* $h_W^{0,2}(X)$ *is odd then the dimension of* $\operatorname{Im} d \subseteq H^0(X,\Omega_X^2)$ *is strictly smaller than* $h_W^{0,2}$.

Remark. i) As will be seen in the proof this dimension is always less than or equal to $h_W^{0,2}$ so we exclude the possibility of an equality.

ii) In Nygaard's case $h_W^{0,2} = 1$ so his conclusion is that all the 1-forms are closed.

Proof : By duality we need to prove that the codimension of $\operatorname{Ker} d$ in $H^2(X,\mathcal{O}_X)$ is less than $h_W^{0,2}$. The image of $V^{-\infty}ZH^2(X,W\mathcal{O}_X)$ in $H^2(X,\mathcal{O}_X)$ lies in the kernel so this codimension equals the codimension of $\operatorname{Ker} d$: $(H^2(X,W\mathcal{O}_X)/V^{-\infty}ZH^2(X,W\mathcal{O}_X))/VH^2(X,W\mathcal{O}_X) \longrightarrow H^2(X,W\Omega_X^1)/dVH^2(X,W\mathcal{O}_X) + VH^2(X,W\Omega_X^1)$ in $(H^2(X,W\mathcal{O}_X)/V^{-\infty}ZH^2(X,W\mathcal{O}_X))/VH^2(X,W\mathcal{O}_X)$, as $R_1 \otimes_R^L R\Gamma(W\Omega_X^\cdot) = R\Gamma(\Omega_X^\cdot)$ (cf. [Il-Ra]) , which has dimension $h_W^{0,2}$ as $m^{0,2} = 0$ by the condition on the Néron-Severi group. Hence if the theorem is false $\operatorname{Ker} d = 0$ and so if σ is the type of $\operatorname{dom}^0(H^2(X,W\mathcal{O}_X) \longrightarrow H^2(X,WS_X^1))$ we get from (III:3.3) that

$\sigma(i)=0$ if $i\in\mathbb{Z}_+$ However as $NS\otimes W=H^1(X,W\Omega^1)$ we get from (III:3.3) that $\mathrm{ord}_p\mathrm{disc}\ NS/\mathrm{tors}=2\sum_{i\in\mathbb{N}} i\sigma(i)$ and by $\sigma(i)=0$ $\forall i\in\mathbb{Z}_+$ we get $p\nmid\mathrm{disc}\ NS/$ tors and as $NS\otimes\mathbb{Z}_\ell=H^2(X_{\bar{k}},\mathbb{Z}_\ell(1))$ we get by ℓ-adic Poincaré duality that NS/tors is unimodular. By the R-R theorem $KF\equiv F^2$ (2) for every $F\in NS$ and so by ([Se:V:Thm 2]) $K^2\equiv\tau(NS/\mathrm{tors})$ (8) where $\tau(-)$ is the index but by Hodge's index theorem $\tau(NS/\mathrm{tors})=-b_2+2$. Noether's formula gives $12\chi=c_2+K^2\equiv-b_2+2+c_2=4+4h_W^{0,1}$ (8) and together with Crew's formula (cf. (IV:3.2)) this gives $1-h_W^{0,1}+h_W^{0,2}\equiv1+h_W^{0,1}$ (2) i.e. $h_W^{0,2}\equiv0$ (2) contrary to assumption.

VII

Complements

I will quickly sketch some possible extensions of the theory.

1. One could, as in [Ny], study $R_n \otimes^L_R M$, $M \in D^b_c(R)$, instead of $R_1 \otimes^L_R M$. If we put $h_n^{i,j}(M) := \mathrm{lgth}\, H^j(R_n \otimes^L_R M)^i$ then we get, with the same proof as for $n = 1$, $\sum_i (-1)^j h_n^{i,j} = n \sum (-1)^j h_W^{i,j}$ and so $nh_W^{i,j} \leq h_n^{i,j}$. From this follows that the n-Hodge polygon, $\mathrm{Hodge}(h_n^{\cdot,m-\cdot})$ lies below the Hodge-Witt. In case $nb_m(M) = \sum_{i+j=m} h_n^{i,j}$ we get the expected result, obtained by Nygaard in [Ny] in the geometric case. This could be proved by redoing our argument for arbitrary n or by paraphrasing Nygaard's argument. (The last thing could of course also have been done for $n = 1$.) Of course even if we paraphrased Nygaard's argument we would have to do it for $\widetilde{H}^m(M)$ and not for M.

2. We have not treated the conjugate spectral sequences. They can in fact be defined for any $M \in D^b_c(R)$. For $n = 1$ we have the sequence of k-complexes $\cdots \longrightarrow \tau_{\leq i}\, \mathrm{Hodge}(M) \longrightarrow \tau_{\leq i+1}\, \mathrm{Hodge}(M) \longrightarrow \cdots$ and the spectral sequence associated to such a "filtration" is the conjugate spectral sequence of level 1 of M. The generalization to arbitrary levels is immediate and then we pass to the limit as in [Il-Ra] and, again as in loc. cit., we define F' and V'. We can then prove the coherence of the E^2-term by dévissage and explicit computation or paraphrasing loc. cit. One can then with the methods used in this paper prove analogues to the results obtained on the slope spectral sequence.

3. If one wants to continue the study of the spectral sequence for $M \in D_c^b(R)$

$$E_1^{i,j} = H^j(M)^i \Longrightarrow H^{i+j}(\underline{\underline{s}}(M))$$

one should note that it is really the degree ∞ part of a spectral sequence of F-gauge structures

$$(3.1) \qquad E_1^{i,j} = S(H^j(M)^i) \Longrightarrow H^{i+j}(\underline{\underline{S}}(M)) \; .$$

Note that, from the E_2-term on, it is a spectral sequence of coherent F-gauge structures and is the spectral sequence associated to the cohomological functor $H^0(\underline{\underline{S}}(-))$ on $D_c^b(R)$ with its standard t-structure. Furthermore, we can consider this spectral sequence as a spectral sequence in G and so of R-complexes. As such it is the spectral sequence associated to the cohomological functor $H_g^0(-)$ on $D_c^b(R)$ with the standard t-structure. Note finally that even though the standard t-structure on $D_c^b(R)$ commutes with the diagonal t-structure and the diagonal t-structure commutes with the F-gauge t-structure, $(G^{\leq 0}, G^{\geq 0})$, the standard t-structure does not commute with the F-gauge t-structure, because if it did, by the description just given, (3.1) would degenerate at E_2 which it does not always do (cf. [Ek 1:V.2]). Hence commutation among t-structures is not a transitive relation.

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L'auteur est un géomètre algébriste suédois qui a obtenu son PhD en mai 1983, à Göteborg. Le texte résout des questions difficiles sur la cohomologie cristalline des variétés propres et lisses sur un corps parfait de caractéristique > 0, telles que l'interprétation de la filtration aboutissement de la suite spectrale des pentes et la reconstruction du terme E_1 à partir de l'action de F sur l'aboutissement, questions que les spécialistes n'avaient qu'effleurées et auxquelles l'auteur apporte des réponses complètes. En même temps, il introduit plusieurs notions originales qui donnent un éclairage nouveau sur cette théorie : (i) décomposition diagonale des complexes R-modules ($R =$ l'anneau de "Cartier-Dieudonné-Raynaud"),(ii) F-structures de jauge (une généralisation naturelle au niveau $\geqslant 1$ de la notion de module de Dieudonné, qui joue un rôle important dans les derniers travaux de Fontaine, Kato et Messing sur la comparaison entre cohomologie étale p-adique et cohomologie de De Rham des variétés propres et lisses en inégale caractéristique),(iii) les nombres de Hodge-Witt, qui permettent un contrôle "polygonal" très fin de la suite spectracle des pentes. Enfin, l'auteur illustre son formalisme par de jolis exemples géométriques, avec notamment une étude détaillée de la "pathologie" des surfaces de Zariski.

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